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Modules which are invariant under automorphisms of their covers and envelopes

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To my parents

Introduction

Projective and injective resolutions are well known tools in Algebra. It is natural to ask whether we can take other types of resolutions and how one can define a generalized concept of cover and envelope. We will present here an approach due to P. A. Guil Asensio, D. K. Tütüncü and A. K. Srivastava [12] (Chapter 1). It consists in defining a notion of χ-envelope where χ is a class of modules closed under isomorphisms. The central topic of this work is then to study the properties of modules which are invariant under automorphisms of their χ-envelopes and covers. We will then apply the results to the special cases in which the class χ is that of the injectives or the projectives.

The problem of studying which modules are invariant under endomorphism of their envelopes can be tracked back to Johnson and Wong [15], who discovered that endomorphism-invariant modules coincide with the quasi-injective ones. Later, Dickson and Fuller [4] studied modules invariant under other subsets of the endomorphisms ring of their injective envelope, mainly the subset of automorphisms. In order to generalize these concepts, we will introduce in the first chapter the notion of χ-envelope following [12], and in the second chapter we will explore them in the setting of χ-envelopes [12]. We will then see, following [11], that the problem of studying whether a χ-automorphism-invariant module M is χ-endomorphism invariant is linked to the study of the unit structure of the endomorphism ring of its χ-envelope modulo the Jacobson radical.

In the second chapter of this thesis, we will also follow the presentation of A. Facchini[6] in order to prove that every module has a pure-injective envelope and we will find out that, as for injective modules, the endomorphism ring of a pure-injective module M has the property of the lifting of idempotents modulo the Jacobson radical and that $\mathrm{End}(M)/J(\mathrm{End}(M))$ is Von Neumann regular right self-injective ([7], [14]).

Pushed by the behaviour of the endomorphism rings of pure-injective and injective modules, and moved by the problem of determining whether a χ-automorphism invariant module is χ-endomorphism invariant, it will be then natural to study the unit structure of Von Neumann regular right self-injective rings. For this type of rings we will consider a decomposition due to P. A. Guil Asensio, T. C. Quynh and A. K. Srivastava [11] into a Boolean ring and a ring in which every element is the sum of two units.

In Chapter 3, we will collect the results of the previous chapters in orther to study when a χ-automorphism-invariant module is χ-endomorphism invariant supposing that the endomorphism ring of the χ-envelope of the module has the natural properties discovered to hold for pure-injective and quasi-injective modules (P. A. Guil Asensio, D. K. Tütüncü and A. K. Srivastava [12]).

In Chapter 4, we will follow [11] to apply the results obtained in Chapter 3 to the special case in which χ is the class of injective modules, and we will see in this case that χ-automorphism invariant modules coincide with the pseudo-injective ones. In particular, we find out ([11], [1]) that for commutative noetherian rings and $\mathbb{F}$-algebras, where $\mathbb{F}$ is a field with more than two elements, quasi-injective modules coincide with the automorphism-invariant ones. We will also see that in general this equivalence does not hold [22].

In Chapter 5, following [12], we will do for covers what we did for envelopes paying attention to the fact that not every module has a projective cover, and this forces us to introduce perfect rings. It is worth noticing that in order to obatin the equivalence between pseudo-projective and automorphism-coinvariant modules (when possible), we have to find the counterpart of a theorem for injectives that states that a module M is automorphism-invariant if and only if every isomorphism between two essential submodules of M can be extended to an endomorphism of M. The key to extend this result is represented by dual automorphism-invariant modules [21].

Contents

Chapter 1

Preliminaries

Throughout this paper, all rings are assumed to be associative with zero element different from the unit, and all modules are assumed to be unitary.

When the side of a module is not specified, we will assume we are talking about right modules. When not specified, a morphism between two modules will be a right module morphism.

For a right R-module M, we usually write $\mathrm{End}(M)$ to indicate its ring of right R-module endomorphisms (we'll write $\mathrm{End}(M_R)$ if there is some ambiguity).

We will say that S is a *subring* of a ring R if $(S, +)$ is a subgroup of $(R, +)$, $(S, \cdot)$ is a subsemigroup of $(R, \cdot)$, and S is a ring; we say that a subring S is a *unitary subring* of R if $(S, \cdot)$ is a submonoid of $(R, \cdot)$.

The right annihilator of a module M will be indicate by $r(M)$ and the right annihilator of an element m of M will be denoted by $r(m)$.

Let M be a right R-module, and let N be a submodule of M; the module N is said *fully invariant* with respect to M if for every endomorphism f of M we have that $f(N) \subseteq N$. The intersection of the maximal submodules of M is called the *Jacobson radical* of M and can be characterized as the intersection of all kernels of morphisms from M to simple modules (it is a fully invariant submodule); we will denote it with the symbol $J(M)$. If $J(M) = 0$ the module M is said to be *semiprimitive*; so clearly $M/J(M)$ is semiprimitive for every module M. If R is a ring and $J(R)$ is a maximal ideal we say that R is a *local* ring.

1.1 Artin-Wedderburn Theorem

For completeness, we recall here the *Artin-Wedderburn Theorem*, which gives a complete characterization of semisimple rings.

Theorem 1.1 (Artin-Wedderburn). *For a ring R, the following are equivalent:*

(a) The ring R is semisimple as a right (left) module over itself.

(b) Each right (left) module over R is semisimple.

(c) Each right (left) module over R is projective.

(d) The ring R is a finite product of simple artinian rings.

(e) The ring R is a finite product of matrix rings over division rings.

1.2 Essential, small and closed submodules

In this section, we recall the notions of *essential* and *small* (*superfluous*) submodules and we will introduce the notion of *closed* submodules.

A submodule N of a right R-module M is said to be *small* (or *superfluous*) in M if whenever there exists a submodule H in M such that $N + H = M$ then $H = M$. By Nakayama's Lemma, we know that for a finitely generated right R-module M the submodule $MJ(R)$ is small in M. We say that an epimorphism $g : M \to N$ is *small* (or *superfluous*) if $Ker(g)$ si small in M. It's easy to prove that if $h : L \to M$ is such that $g \circ h$ is an epimorphism than h is also an epimorphism (this simple remark turns out to be useful in the study of projective covers).

A submodule N of M is said to be *essential* in M if for every non-zero submodule H of M we have that $N \cap H \neq 0$ and we write $N \leq_e M$ (we say then that M is an *essential extension* of N). Similarly to what we have done before, we say that a monomorphism $f : M \to L$ is *essential* if $f(M)$ is an essential submodule of L. It is easy to show that if $h : L \to X$ is a morphism such that $h \circ f$ is injective then h must be injective. This notion of "density" is the main tool to define an injective envelope (an essential morphism $f : M \to L$ can be thought as a morphism with an image spreading all over the submodules of L, and one could wonder how a maximal essential extension can be characterized).

We just recall for completeness some properties of essential submodules:

Proposition 1.2. *The following hold:*

(a) If $K \leq N \leq M$, then $K \leq_e M$ if and only if $K \leq_e N$ and $N \leq_e M$.

(b) If N, $N' \leq M$, then $N \cap N' \leq_e M$ if and only if N and N' are essential in M.

(c) If $f : M \to N$ is a morphism and $N' \leq_e N$ then $f^{-1}(N') \leq_e M$.

(d) If $N_1 \leq M_1$ and $N_2 \leq M_2$, then $N_1 \oplus N_2 \leq_e M_1 \oplus M_2$ if and only if N_1 and N_2 are essential respectively in M_1 and M_2.

(e) More generally, if $M_i \leq_e X_i$ for every $i \in I$, then $\oplus_{i\in I} X_i \leq_e \oplus_{i\in I} M_i$.

(f) If N is an essential submodule of M, then for every module X in M we have that $N \cap X$ is essential in X.

Suppose now that M is a not a cyclic right R-module and consider an element $m \in M$; by hypotesis $mR \neq M$. If there is not a non-zero element $m_1 \in M$ such that $m_1 R \cap mR = 0$, then M is an essential extension of mR. Otherwise, we can consider $M_1 = m_1 R \oplus mR$ and we can try to find a non-zero element m_2 such that $m_2 R \cap M_1 = 0$. Proceding in this fashion it is easy to show that M is an essential extension of a direct sum of some of its cyclic submodules.

A submodule N of M is said to be *closed* in M if N coincides with any submodule of M which is an essential extension of N. If $\bar{N}$ is a closed essential extension of N, with $\bar{N} \leq M$, we say that $\bar{N}$ is *a closure* for N in M.

A natural question arises: can we find for every submodule $N \subseteq M$ a submodule X such that: $N \cap X = 0$, $X \oplus N \leq_e M$ and $Y \cap N \neq 0$ for every Y with the property $X \leq Y$?

The answer in affirmative:

If N is essential, there is nothing to prove, so we can suppose it is not essential. Consider then the set I whose elements are the modules $C \leq M$ such that $N \cap C = 0$, and order I by inclusion. It is easy to see that we can apply Zorn's Lemma, and so there exists a maximal element $X \in I$. The element X satisfies all the properties we are looking for; in fact $N \cap X = 0$, and if Y is a submodule of M which properly contains X, we have that $Y \cap N \neq 0$.

Notice that X is closed: consider the set J of all the submodules H of M which contain X and such that X is essential in H. If J is not empty, ordering J by set inclusion, satisfies Zorn's Lemma. If we take a maximal element $\bar{X}$ of J, then we have that $\bar{X} \cap N = 0$, so that $X = \bar{X}$.

We call X a $\cap$-*complement* of N in M. In particular, we showed that every submodule of a module M has a $\cap$-complement, so in particular every submodule of M is a direct summand of an essential submodule of M.

It is easy to notice that if N and X are two submodules of M such that $N \cap X = 0$, then there exists a closed submodule $\bar{X}$ such that $X \subseteq \bar{X}$ and $\bar{X}$ is a $\cap$-complement of N.

From the previous considerations, it follows that a $\cap$-complement is closed. Is the converse true? Is every closed a $\cap$-complement?

If N is a closed submodule of M, we know that it has a $\cap$-complement X, and one can show that N is a $\cap$-complement of X (it suffices to use the notion of essential extension). So we can conclude that the class of $\cap$-complements in M coincides with the class of closed submodules of M.

We will see in the next sections that the notion of $\cap$-complement is deeply linked with the problem of decomposing the injective envelope of a module.

1.3 Injective modules and injective envelopes

In this section, we will introduce the concepts of injective modules, injective envelopes and we will introduce also the notion of "uniform module". Injective modules and injective envelopes will be a fundamental tool throughout this work.

We say that a a right R-module E_R is *injective* if for every submodule M'_R of a right R-module M_R we have that every morphism $f : M'_R \to E_R$ can be extended to a morphism from M to E_R.

Now we can use the notion of proper essential extension we have introduced in Section 1.2 to define what is an injective envelope of a module; the goal is to construct an essential extension E of a module M such that E is injective.

An injective envelope of a module M is a couple (E, i) where E is an injective module and $i : M \to E$ is an essential monomorphism.

One can show the following properties:

Theorem 1.3 (Fundamental Lemma of injective envelopes)**.** *Let (E, i) be an injective envelope of a module M and F an injective module. Suppose there is a monomorphism $j : M \to F$; then F has a direct sum decomposition $F = F' \oplus F''$ where $j(M) \subseteq F'$, $F' \cong E$ and if $j' : M \to F'$ is the composite of j and the projection on F' we have that (F', j') is an injective envelope of M.*

Theorem 1.4. *Every module has an injective envelope which is unique up to isomorphism. In particular, if (E, i) and (E', i') are two injective envelopes of a module M, then there exists an isomorphism $h : E \to E'$ such that $h \circ i = i'$.*

While the previous theorems show that injective envelopes are minimal injective extensions, the following theorem shows that injective envelopes can be also thought as maximal essential extensions.

Theorem 1.5. *An extension (E, i) of a module M is an injective envelope if and only if it is a maximal essential extension of M. In other words: (E, i) is an injective envelope of M if and only if and only if it is an essential extension of M, and for every morphism $f : E \to N$ such that $f \circ i$ is an essential extension of M we have that f is an isomorphism.*

1.4 Uniform modules

In this section, we introduce the concepts of uniform modules and uniform dimension; we will indicate the envelope of a module M with the symbol $E(M)$, not specifying the monomorphism.

The notion of uniform module is related to the fact that if $N \leq M$, where N and M are two modules, then we have that $E(N)$ is a direct summand of $E(M)$. One could ask: which are the modules M such that $E(M)$ is indecomposable?

Notice that if $E(M)$ is indecomposable and N and N' are two submodules of M such that $N \cap N' = 0$, then we have that $E(M) = E(N) \oplus E(N') \oplus C$, for some submodule C of $E(M)$. But if $E(M)$ is indecomposable this means that at least one of $E(N)$ and $E(N')$ is zero, and so $N = 0$ or $N' = 0$. We say that a non-zero module M is *uniform* if for every two non-zero submodules N and N' of M we have that $N \cap N' \neq 0$.

In particular we can prove the following:

Theorem 1.6. *The following are equivalent for a non-zero module M:*

(a) If N, N' are two submodules of M such that $N \cap N' = 0$ then we have that $N = 0$ or $N' = 0$.

(b) M is uniform.

(c) Every non-zero submodule of M is essential in M.

(d) The injective envelope of M is indecomposable

Proof. The only non-trivial part that remains to be proved is "(a) $\Rightarrow$ (d)". Suppose that $E(M) = E_1 \oplus E_2$ and that both E_1 and E_2 are non-zero modules. Then we have that $M \cap E_i \neq 0$ with $i = 1, 2$. But this is absurd since M is uniform and $(E_1 \cap M) \cap (E_2 \cap M) = 0$. □

We can now show the following:

Theorem 1.7. *For a non-zero injective module E the following are equivalent:*

(a) The module E is indecomposable.

(b) The module E is uniform.

(c) The endomorphism ring of E is local.

Proof. (a)⇒ (b) is immediate by 1.6 after noticing that the injective evelope of an injective module is the module itself.

(c) ⇒ (a) is immediate.

(b) ⇒ (c) To show that the endomprhism ring of E is local it suffices to show that for every two non invertible elements of $\mathrm{End}(E_R)$ the sum is again non invertible. Let's start by proving that a monomorphism $f : E \to E$ is an automorphism. We know that $f(E)$ is an injective module, and so it is a direct summand of E; but since E is indecomposable by hypotesis, it follows that f is surjective. If ϕ and ψ are two non invertible morphisms of $\mathrm{End}(E_R)$ then we have that ϕ and ψ are not monomorphisms (otherwise they would be automorphisms by the previous consideration); then $ker(\phi) \cap ker(\psi) \neq 0$ since E is uniform and so $\phi + \psi$ is not invertible. □

Example 1.4.1. It is obvious that over semisimple rings a module is uniform if and only if it is simple; it is also true that a module over a semisimple ring is uniform if and only if it is indecomposable (Artin-Wedderburn Theorem 5.25). However the next situations will show that these condituons are not equivalent:

(a) Let $R = \mathbb{Z}$, and let $M = \mathbb{Z}$. Then M is clearly uniform but not simple.

(b) Let $R = \mathbb{Q}$ and let $\mathbb{Q}[u, v]$ be the commutative R-algebra defined by the relations $u^2 = v^2 = uv = 0$. Clearly R is indecomposable as a right R-module, but it is not uniform since its submodules $\mathbb{Q}[v]$ and $\mathbb{Q}[u]$ do not intersect.

1.5 Uniform dimension

In this section, we introduce the concept of *uniform* (or *Goldie*) dimension of a module. This notion was introduced by Alfred Goldie to provide a concept of dimension for modules generalizing the concept of dimension of a vector space.

We say that a family of submodules $\{N_i \mid i \in I\}$ of a module M is *independent* if its sum ΣN_i is direct. Everytime we have such a family an easy consequence of Zorn's Lemma is that this family is contained into a maximal independent one.

We would like to study this notion in correlation with uniform modules.

Lemma 1.8. *Let M be a non-zero module without uniform submodules. Then M contains an infinite independent family of non-zero submodules.*

Proof. By induction, we want to construct a family $\{N_1, N_2, \dots\}$ of non-zero independent submodules of M such that for every index n the module $N_1 \oplus \cdots \oplus N_n$ is not essential in M. For $n = 1$, since M is not uniform, we know that there exist N_1, N_1' non-zero submodules of M such that $N_1 \cap N_1' \neq 0$, and clearly N_1 is not essential in M. Suppose now we have constructed a independent family $N_1, \dots N_n$ such that their direct sum is not essential in M. So there exits $B \leq M$ such that $B \cap N_1 \oplus \cdots \oplus N_n = 0$. Since B is not uniform there exist two non-zero submodules N_{n+1} and N'_{n+1} of B such that $N_{n+1} \cap N'_{n+1} = 0$. We have that $N_1 \oplus \cdots \oplus N_{n+1}$ is not essential since its intersection with N'_{n+1} is zero. □

Theorem 1.9. *For a non-zero module M the following are equivalent:*

(a) The module M_R does not contain an infinite independent family of non-zero submodules.

(b) The module M contains a finite independent set of uniform submodules $\{N_1, \dots, N_n\}$ such that $N_1 \oplus \cdots \oplus N_n$ is essential in M.

(c) There exists an integer m such that each indipendent family of non-zero submodules of M has cardinality less than m.

(d) If we have an increasing sequence $N_1 \leq N_2 \leq \dots$ of submodules of M, then there exits an $i \geq 0$ such that N_i is essential in N_j for every $j \geq i$ (it is equivalent to say that we have the maximum condition on closed submodules).

Proof. (a)⇒ (b) From Lemma 1.8 we know that M must contain a maximal independent family of uniform submodules $J = \{N_i \mid i \in I\}$, and by hypotesis this family is finite (suppose $I = \{1, \dots, n\}$). We just need to show that $H = N_1 \oplus \cdots \oplus N_n$ is essential in M. Suppose then that there exists $B \leq M$ such that $B \cap H = 0$. By Lemma 1.8, we have that B contains a uniform submodule, against the maximality of the family J.

(b) ⇒ (c) Consider a family of modules $X = \{N_1, \dots, N_n\}$ in M of cardinality n such that (b) holds. We want to show that every family of non-zero independent submodules of M has at most cardinality n. Let's act by contraddiction and suppose there exists a set $Y = \{B_1, \dots, B_k\}$ of non-zero indipendent submodules of M such that $n \leq k$. We will show that for every $t = 0, \dots, n$ there exist subsets X_t of X of cardinality t and Y_t of Y of cardinality $k - t$ such that $X_t \cap Y_t = \emptyset$ and $X_t \cup Y_t$ is an indipendent family of submodules fo M; suppose we have proved the previous statement. Then, considering X_n and Y_n, since $k - n \geq 0$, we have that $N = N_1 \oplus \cdots \oplus N_n$ is not essential in M, a contraddiction.

The case $t = 0$ is trivial. Suppose now we have constructed X_t and Y_t and we want to construct X_{t+1} and Y_{t+1}. Since $k - t \geq n - t > 0$, Y_t is non-empty and we can consider the module $C = (\oplus_{N_i \in X_t} N_i) \oplus (\oplus_{B \in Y_t \setminus \{B_j\}} B)$. If $C_i = C \cap N_i \neq 0$ for every i, then C_i is essential in every N_i and this implies that $D = \oplus C_i$ is essential in N. Since C contains D, C is essential in M, against the hypotesis. This implies that there exists an index l such that $C \cap N_l \neq 0$. Considering now $X_{t+1} = X_t \cup \{N_l\}$ and $Y_{t+1} = Y_t \setminus \{B_j\}$ we can conclude.

(c) $\Rightarrow$ (d) If (d) is not satisfied then we have an increasing sequence of type $N_1 \leq N_2 \dots$ such that for every i there exits a $j(i)$ such that N_i is not essential in $N_{j(i)}$. So consider $B_0 = N_0$, then there exists $B_1 \leq N_{j(0)}$ such that $B_0 \cap B_1 = 0$. It suffices to proceed in this fashion in order to obtain an infinite indipendent family of non-zero submodules of M.

(d)$\Rightarrow$ (a) If (a) does not hold then we have an infinite independent family of modules $\{B_i \mid i \in I\}$. It suffices to consider the modules $N_i = \oplus_{i=0}^{n} B_i$ to obtain a contraddiction.

□

It is important to notice that, in Theorem 1.9, we have shown something more than what is stated: if a module M contains a finite independent set of cardinality n of uniform submodules such that their direct sum is essential in M, then every other independent family of non-zero submodules of M has cardinality $\leq n$. These considerations make us able to define what is known as the *Goldie dimension* of a module. We say that a module M has *infinite uniform dimension* (or *infinite Goldie dimension*) if it contains an infinite independent set of non-zero submodules, otherwise we say that the module has *finite uniform dimension*; in particular, when a module M has finite uniform dimension, we say that its dimension is n if the cardinality of a maximal set of independent uniform submodules of M is n. We indicate the dimension of M by $dim(M)$, and since M in essential in $E(M)$ we have that $dim(M) = dim(E(M))$. Hence, when $dim(M) = n$, M contains a finite direct sum of uniform submodules which is essential in M, and we can write $E(M) = E(N_1) \oplus \cdots \oplus E(N_n)$ where $E_1, \ldots, E_n$ are indecomposable (and so uniform) modules. We can say then, that a module M has uniform dimension n if and only if $E(M)$ is a direct sum of n non-zero indecomposable modules.

It is worth saying that the equivalent properties of Theorem 1.9 are equivalent to ask M to have the minimum condition on closed submodules. In particular noetherian and artinian modules have finite uniform dimension.

We can say something more about the dimension of an artinian module. In fact we know that the socle of a module is just the intersection of all the essential ideals of the module. If M is artinian then its socle is essential in M, so we get that $dim(M) = dim(\text{soc}(M))$ which is just the composition length of $\text{soc}(M)$.

1.6 Local modules and semiperfect rings

In this section, we will see how the notion of being local can be extended to modules and we will define semiperfect rings.

A module M is said to be *local* if it is cyclic and $M/J(M)$ is a simple module.

One can prove that if we take and idempotent element e of a ring R then the following are equivalent:

(a) The ring eRe is local.

(b) The module eR_R is local.

(c) The module ${}_RRe$ is local.

We say in this case that e is *local*.

We need also to define *semiperfect rings*, in fact this class of rings will be deeply connected with the study of exchange modules with a finite direct sum decomposition into indecomposables.

A ring R is said to be *semiperfect* if it satisfies one of the following equivalent conditions:

(a) The module R_R is a finite direct sum of local modules.

(b) The module ${}_RR$ is a finite direct sum of local modules.

(c) The unity element of R is the sum of finitely many ortoghonal local idempotents.

(d) The ring $R/J(R)$ is semisimple and if $x + J(R)$ is an idempotent of $R/J(R)$ there exists an idempotent element $e \in R$ such that $e - x \in J(R)$ (this means that *idempotents lift modulo* $J(R)$, but we will give an accurate definition later on).

1.7 Properties of injective and projective modules

In this section, we will give a more general notion of injectivity and projectivity for a module (Anderson and Fuller [2] pp. 184-185).

Let's fix a module M and consider a module N. We will define a concept of M-injectivity analogous to the usual concept of injectivity we know. So we say that N is M-injective if for every submodule M' of M, every morphism $f : M' \to N$ can be extended to a morphism from N to M. We say that a module is *quasi-injective* if it is injective with respect to itself.

We can proceed in an analogous way to generalize the definition of "projective modules". We say that a module M is projective with respect to the module N (or N-projective) if for every epimorphism $g : N \to \bar{N}$ and every morphism $f : M \to \bar{N}$ there exists a morphism $h : M \to N$ such that $g \circ h = f$. We say that a module is *quasi-projective* if it is projective with respect to itself.

This kind of approach leads us to generalize some properties of injective and projective modules to other classes of modules.

Theorem 1.10. *Consider a module M. The class χ consisting of the modules N such that M is N-injective is closed for homomorphic images, direct sums and submodules.*

The class χ consisting of the modules N such that M is N-projective is closed under submodules, homomorphic images and finite direct sums.

Proof. We will just prove the first part of the theorem. The fact that χ is closed for submodules its obvious. Suppose $N \in \chi$ and let $\bar{N}$ be an homomorphic image of N through the application g, and $\bar{N}'$ be a submodule of $\bar{N}$. Call N' the preimage of $\bar{N}'$ through g and consider a morphism $f : \bar{N}' \to M$. Then there exists a morphism $h : N \to M$ such that $f \circ g_{|N'} = h_{|N'}$. But N' contains the kernel of g, then $ker(g) \subseteq ker(h)$ and so we can lift h to an application $\bar{h}$ from $\bar{N}$ to M (and $\bar{h}$ satisfies the requested properties).

Suppose now that M is an N_i-injective module for every $i \in I$ and consider $N = \oplus_{i \in I} N_i$. We want to show that M is N-injective. Suppose that we have a submodule N_1 of N and a morphism $f : N_1 \to M$. Consider now the set S

consisting of the couples (h, L) such that $N \leq L$ and h extends f. We can order the couples in this way : $(h, L) \leq (h_1, X)$ if $L \subseteq X$ and h_1 extends h. We can apply Zorn's Lemma to the preordered set S, so that we can choose a maximal element $(\bar{h}, \bar{N})$. It suffices now to show that $\bar{N} = N$. Notice that $\bar{h}$ resctricted to $\bar{N} \cap N_i$ can be extended to an application $h_i : N_i \to M$. If we consider the application $v : N_i + \bar{N} \to M$, defined by $v(x+y) = h_i(x) + \bar{h}(y)$, it is easy to check that it is well defined and it is a morphism (if $x + y = 0$ then $x = -y$ so that $v(x + y) = \bar{h}(x) + \bar{h}(-x) = 0$). Therefore, every N_i is contained in $\bar{N}$ and we can conclude.

□

We say that a ring R is *right self-injective* if it is injective as a right R-module. From Theorem 1.10, we can obtain almost immediately Baer's criterion, but we can say also a little bit more:

Theorem 1.11. *If M_R is an injective module with respect to the module N_R, and we assume that N_R contains an isomorphic copy of the module R_R, then M is injective. In particular, R is right self-injective if and only if R is right quasi-injective.*

Proof. Notice that by Theorem 1.10 we have that M is R-injective. The proof can be obtained in two ways. The first is two use Baer's criterion, the other one is to use Theorem 1.10 noticing that M is injective with respect to every direct sum of copies of R_R, and so it is injective with respect to every homomorphic image of direct sums of R_R. But since every module is an homomorphic image of a direct sum of copies of R_R we can conclude. □

Now, we want to find now an analogue of some well known properties of injective and projective modules in our new framework. In particular, we know that every direct summand of an injective (resp. projective) is injective (resp. projective) and that a direct product (resp. direct sum) of injectives (resp projectives) is injective (resp. projective). Can we find something similar for N-injective modules?

The following theorem shows that the answer is positive:

Theorem 1.12. *Consider a module N. A direct product $M = \Pi_{i \in I} M_i$ is N-injective if and only if each M_i is N-injective. A direct summand of an N-injective module is N-injective.*

A direct sum $\oplus_{i \in I} M_i$ is N-projective if and only if each M_i is N-projective. A direct summand of an N-projective is N-projective.

Proof. We prove just the first part of the theorem since the second part can be proved similarly. We indicate the projections of M on the M_i and the inclusions of the M_i's in M respectively by π_i and ι_i.

Suppose that M is N-injective and consider a submodule H of N and a morphism $f : H \to M_i$. The application $\iota_i \circ f$ can be extended to a morphism $\bar{f} : N \to M$. It is easy to see that the morphism we are looking for is $\pi_i \circ \bar{f}$.

Suppose now that all the modules M_i are N-injective. Consider a submodule H of N and a morphism $f : H \to M$. Then every $\pi_i \circ f$ can be extended to a morphism f_i from N to M_i. By the Universal Property of Direct Products, there exists $\bar{g} : N \to M$ such that $\pi_i \bar{g} = f_i$. It is easy to check that $\bar{g}$ extends f.

It remains to prove the result for direct summands. Suppose that A is a direct summand of M, where M is an N-injective module, suppose that we have a morphism $f : H \to A$, where H is a submodule of N, and call ι_A the inclusion of A in M. Then $\iota_A \circ f$ can be extended to a morphism g from N to M. It is easy to check that the morphism we are looking for is $\pi_A \circ g$.

□

We know that if a module M has an injective submodule E, then E is a direct summand of M. Can we extend this property also to N-injective modules?

Theorem 1.13. *Suppose that M is an N-injective module and that there exists a monomorphism $f : M \to N$. Then M is isomorphic to a direct summand of N and is quasi-injective.*

In particular, if N is indecomposable, or if f is essential, then f is an isomorphism.

Proof. We have that $M \cong f(M)$ and since M is N-injective than $f(M)$ is quasi-injective (theorem 1.10); so there exists a morphism $g : N \to f(M)$ such that $g \circ \iota_{|f(M)} = 1_{|f(M)}$. This implies that $ker(g) \cap f(M) = 0$. As for every $n \in N$ we have that $n = n - (g(n)) + (g(n))$, we obtain that $f(M) \oplus ker(g) = M$. □

An analogous statement holds for N-projective modules:

Theorem 1.14. *Let M be an N-projective module and suppose there is an epimorphism $g : N \to M$. Then M is isomorphic to a direct summand of N and therefore it is quasi-projective. If the module N is indecomposable or the epimorphism g is small, we have that g is an isomorphism.*

As an immediate consequence we obtain the following:

Theorem 1.15. *If N' is a submodule of N and N/N' is N-projective, then N' is a direct summand of N. In particular the cyclic module xR is projective if and only if there exists an idempotent $e \in R$ such that $r(x) = eR$ (if and only if $r(x)$ is a direct summand of R_R).*

Example 1.7.1. An example of a $\mathbb{Z}$-module that is quasi-projective but not projective is given by $\mathbb{Z}/4\mathbb{Z}$ (it is immediate to verify that $\mathbb{Z}/4\mathbb{Z}$ is not projective but is quasi-projective).

1.8 Injective envelopes and quasi-injective modules

We have introduced in the previous sections quasi-injective modules. Are we able to characterize this concept in another manner? We will prove that quasi-injective modules correspond to *endomorphism-invariant modules.*

We say that a module is *endomorphism-invariant* if it is fully invariant under endomorphisms of its injective envelope.

Let M be an endomorphism-invariant module and let N be a submodule of M. Suppose we we have a morphism $f : N \to M$; we know that this homomorphism can be extended to a morphism $g : E(M) \to E(M)$. But since M is endomorphism-invariant we get that $g_{|M}(M) \subseteq M$, therefore M is quasi-injective. Does the opposite implication hold?

Theorem 1.16. *A module M is quasi-injective if and only if it is endomorphism-invariant.*

Proof. It remains to show just one implication. Suppose that M is quasi-injective and consider an endomorphism f of the injective envelope of M. Consider now the submodule $L = \{m \in M \mid f(m) \in M\}$ of M. Since M is quasi-injective we can extend the morphism $f_{|L}$ to an endomorphism g of M, and g can be extended to a endmorphism h of $E(M)$. We want to show that $h_{|M} = f_{|M}$ by contraddiction. Hence, suppose that $(h - f)(M) \neq 0$; then, since M is essential in $E(M)$, we obtain that $(h - f)(M) \cap (M) \neq 0$. Then there exist m, $m' \in M$ such that $(h - f)(m) = m'$. Then we obtain that $h(m) - m' = f(m)$ and this means that $m \in L$. Therefore $m' = 0$ and M is endomorphism-invariant.

□

It is easy to notice that a module M_R is quasi-injective if and only if M is quasi-injective as a right $R/r(M)$-module.

Example 1.8.1. Let R be a commutative ring, let E be an injective R-module and let I be an ideal in R. Then it is easy to prove that $E[I] = \{m \in E \mid mi = 0 \text{ for every; } i \in I\}$ is quasi-injective. In particular $\mathbb{Z}/p^n\mathbb{Z}$ is quasi-injective but not injective (it is simply $E[p^n\mathbb{Z}]$ where E is the *Prüfer Group* $\mathbb{Z}(p^\infty)$).

We will see during this section that being uniform or being indecomposable are exactly the same thing for a quasi-injective module. So a consequence of theorem 1.7 is that, for a quasi-injective module M, it is equivalent to say that M is indecomposable or that $\text{End}(E(M))$ is local (which is indeed equivalent to say that $E(M)$ is indecomposable).

Lemma 1.17. *Let M be a quasi-injective module and let $E(M) = \oplus_{i\in I}X_i$ be a direct sum decomposition of $E(M)$. Then $M = \oplus'_{i\in I}X'_i$, where $X'_i = X_i \cap M$.*

Proof. Let π_i be the endomorphisms of $E(M)$ that send each element to its projection on X_i. Since M is endomorphism-invariant we get that $\pi_i(M) \subseteq M$ and every element of M has a decomposition of the type $m = \Sigma x_i$ such that $x_i \in X_i$ for every $i \in I$. But then we get that $\pi_i(m) \in X'_i$. So in particular we get that $M = \oplus_{i\in I}X'_i$. □

Using Lemma 1.17, we get the following theorem:

Theorem 1.18. *The following are equivalent for a non-zero right quasi-injective module M:*

(a) The module M is indecomposable.

(b) The module M is uniform.

(c) The ring End(M) is local.

(d) The module $E(M)$ is indecomposable.

(e) The module $E(M)$ is uniform.

(f) The ring End$(E(M))$ is local.

Proof. We already know, from Theorem 1.7, that (b), (d), (e), (f) are equivalent.

Let's show that (a)⇒ (b). Assume that M is not uniform. Then we have that there exits two non-zero submodules A, B of M such that $A \cap B = 0$. Therefore, $E(M) = E(A) \oplus E(B) \oplus C$ and so by Lemma 1.17 we have that $M = A' \oplus B' \oplus C'$ where $A' = M \cap A$, $B' = M \cap B$ and $C = C \cap M$. Since M is indecomposable we have that two between A', B' and C' must be equal to 0. Hence, since M is essential in $E(M)$ we get that two between A, B and C must be equal to 0. So M is uniform.

(f)⇒ (c) By Theorem 1.16, we have a surjective morphism of rings

$$\alpha : \text{End}(E(M)) \to \text{End}(M).$$

Since $\text{End}(E(M))$ is local, then we get that $\text{End}(M)$ is local.

The other implications are easy. □

Now we want to define a class of modules which contains the quasi-injective modules.

We say that a module M is *quasi-continuous* if for every direct sum decomposition of its injective envelope $E(M) = \oplus_{i \in I} X_i$ we have $M = \oplus_{i \in I} X'_i$, where $X'_i = X_i \cap M$.

From Theorem 1.18, we get that every quasi-injective module is quasi-continuous.

There is a deep connection beetween idempotent-invariant modules (modules which are invariant under idempotent endomorphisms of their injective envelope) and quasi-continuous modules. In fact, the following holds:

Theorem 1.19. *The following are equivalent for a right R-module M:*

(a) The module M is quasi-continuous.

(b) The module M is idempotent-invariant.

(c) Every idempotent morphism of each submodule of M can be extended to an endomorphism of $E(M)$.

(d) Every idempotent morphism of each submodule of M can be extended to an idempotent endomorphism of $E(M)$.

(e) Every submodule of M is an essential submodule of some direct summand of M and for any two direct summands X, Y of M with $X \cap Y = 0$ we have that $X \oplus Y$ is a direct summand of M.

(f) For every submodule $X = X_1 \oplus \cdots \oplus X_n$ of M there exists a direct sum decomposition of $M = M_1 \oplus \cdots \oplus M_n \oplus Y$ such that M_i is an essential extension of X_i for every $i = 1, \cdots, n$.

(g) For every submodule $X = M_1 \oplus \cdots \oplus M_n \leq_e M$ such that the M_i are closed in M, we have that $X = M$.

Proof. We will not prove all the implications.

(c)$\Rightarrow$ (b). Let α be an idempotent morphism of the injective envelope of M and consider the set $N = \{m \in M \mid \alpha(m) \in M\}$. The set N is a submodule of M. Consider the resctriction of α to N; by (c) we know it has an extension to an endomorphism f of M, and so to an endomorphism g of $E(M)$. We want to show that $(\alpha - g)(M) = 0$. Suppose that's not the case so that $(\alpha - g)(M) \neq 0$. Since M is essential in $E(M)$ we have that $M \cap (\alpha - g)(M) \neq 0$, so there exist two elements m, m' in M such that $m = (\alpha - g)(m')$. Then we get that $\alpha(m') = (\alpha - g)(m') + g(m') = m + g(m')$ and this means that $m' \in N$, but then $(\alpha - g)(m') = 0$.

(b)$\Rightarrow$ (a). Consider the direct sum decomposition of $E(M)$ given by $E(M) = \oplus_{i\in I} X_i$; we want to show that $M = \oplus_{i\in I} X'_i$ where $X'_i = X_i \cap M$. Consider now the endomorphisms π_i of the injective envelope given by the projection of the injective envelope into its direct summands X_i. Every element $m \in M$ can be written as a sum $m = \Sigma_{i\in I} x_i$. Since the π_i are idempotent we get that $\pi_i(m) \in X'_i$, therefore $M = \oplus_{i\in I} X'_i$. □

One can prove that a right R-module M is quasi-continuous if and only if it is quasi-continuous as a right $R/r(M)$- module.

It is easy to notice that every non-zero submodule of a uniform module is quasi-continuous and uniform (it suffices to apply (f) of Theorem 1.19).

We say that a module M is *continuous* if the following hold:

(a) every submodule of M is essential in a direct summand of M.

(b) every submodule of M which is isomorphic to a direct summand of M is a direct summand of M.

In particular one can show, and it is not difficult, that continuous modules are quasi-continuous. It is also possible to show, but it's not immediate ([18]), that quasi-continuous modules satisfying the finite exchange property, satisfy the exchange property (we will see in the next section what that means).

Theorem 1.20. *The following are equivalent for a right R-module M:*

(a) The module M is continuous.

(b) Every submodule of M isomorphic to a closed submodule of M, is a direct summand of M.

(c) The module M is quasi-continuous and, for every endomorphism f of M such that $ker(f)$ is a closed submodule of M, the submodules $ker(f)$ and $f(M)$ are direct summands of M.

(d) The module M is quasi-continuous and, for every endomorphism f of M, if $ker(f)$ is a direct summand of M, then also $f(M)$ is a direct summand of M.

We can now prove the following :

Theorem 1.21. *Every quasi-injective module is continuous.*

Proof. We want to prove that every submodule of M which is isomorphic to a closed submodule of M is a direct summand of M. To do that consider H, a closed submodule of M, and consider a monomorphism $f : H \to M$; we set $N \equiv f(H)$. Since a quasi-injective is quasi-continuous, by (f) of Theorem 1.19, we get that $M = H \oplus C$. Consider π_H the projection of M onto H and $f^{-1} : N \to H$. Since M is quasi-injective $\iota_H \circ f^{-1}$ can be extendend to an endomorphism g of M. Let $u = f \circ \pi_H \circ g$, then for every $n \in N$ we have that $u(n) = f \circ \pi_h \circ \iota_H \circ f^{-1}(n) = n$. This means that

$$M = Ker(u) \oplus u(N) = Ker(u) \oplus N$$

(for every $m \in M$ we have that $m = m - u(m) + u(m)$). □

Corollary 1.22. *Let M be a quasi-continuous right R-module. Then M is continuous if and only if M has the property that every isomorphism $f : H \to N$, where N is a submodule of M and H is a closed submodule of M, is such that f^{-1} can be extended to an endomorphism of M.*

Proof. See the proof of Theorem 1.21. □

1.9 Exchange modules

In this section we will introduce *exchange* modules. In particular, we will see that quasi-injective modules satisfy the *exchange property* and that a module M_R has the finite exchange property if and only if $R = \text{End}(M_R)$ is an *exchange ring*.

It is known that if A, B and C are submodules of M such that $C \leq A$, then the *modular identity* holds: $A \cap (B + C) = (A \cap B) + C$.

From the modular identity, one obtains that if $M = A \oplus C$ and $A \leq B \leq M$, then $B = A \oplus D$ where $D = B \cap C$.

In fact, we have that $B \cap (C + A) = (B \cap C) + A$.

We say that a module M has the τ-exchange property, where τ is a cardinal, if everytime we have a decomposition $N = M' \oplus C = \oplus_{i \in I} N_i$, where $M \cong M'$ and $|I| \leq \tau$, there exist $N'_i \leq N_i$ such that $N = M' \oplus (\oplus_{i \in I} N'_i)$. Notice that in this case we have $N'_i \leq N_i \leq N'_i \oplus M' \oplus (\oplus_{j \in I, j \neq i} N'_j)$, and by the previous considerations we get that $N_i = N'_i \oplus D$ where

$$D = N_i \cap (M' \oplus (\oplus_{j \in I, j \neq i} N'_j)).$$

So in particular the N'_i's are direct summands of the N_i's.

We say that a module has the *exchange property* if it has the τ-exchange property for every cardinal τ. We say that a module M has the *finite exchange property* if it has the n-exchange property for every finite cardinal n.

Lemma 1.23. *Let G, M', P, N and $(A_i \mid i \in I)$ be modules such that*

$$G = M' \oplus N \oplus P = (\oplus_{i\in I} A_i) \oplus P.$$

Suppose that

$$G/P = (M' + P)/P \oplus (\oplus_{i\in I} B_i + P)/P$$

with $B_i \leq A_i$ for every $i \in I$; then we have that $G = M' \oplus (\oplus_{i\in I} B_i) \oplus P$.

Proof. We get automatically that $M' + \Sigma_{i\in I} B_i + P = M$ and we have just to show that this sum is direct. Suppose that $m + \Sigma b_i + p = 0$ where $m \in M'$, $b_i \in B_i$ and $p \in P$. Then we get that $m \in P$ and $b_i \in P$ for every $i \in I$ and this implies that $m = 0$, $b_i = 0$ for every $i \in I$ and $p = 0$. □

An immediate consequence of Lemma 1.23 is that:

Theorem 1.24. *Let M be a module with the τ-exchange property and let G, M', P , N, $(A_i \mid i \in I)$ be modules such that $|I| \leq \tau$, $M' \cong M$ and*

$$G = M' \oplus N \oplus P = (\oplus_{i\in I} A_i) \oplus P$$

Then there exist $B_i \leq A_i$ such that B_i is a direct summand of A_i for every $i \in I$ and $G = M' \oplus (\oplus_{i\in I} B_i) \oplus P$.

We are now interested in proving that the class of τ-exchange modules is closed under finite direct sums and direct summands.

Theorem 1.25. *Let $M = M_1 \oplus M_2$. Then M has the τ-exchange property if and only if M_1 and M_2 have the τ-exchange property.*

Proof. We prove just one direction. Suppose that M_1 and M_2 have the τ-exchange property and consider $G = M' \oplus N = \oplus_{i\in I} A_i$ where $M' \cong M$ and $M' = M'_1 \oplus M'_2$ where $M'_1 \cong M_1$ and $M'_2 \cong M_2$. Since M_1 has the exchange property we can find $A'_i \leq A_i$ such that

$$G = M'_1 \oplus M'_2 \oplus N = M'_1 \oplus (\oplus A'_i).$$

Since M_2 has the τ-exchange property we can conclude using Theorem 1.24. □

It is pretty simple to show that, a finitely generated module M has the exchange property, if and only if M is a direct summand of a direct sum of finitely generated modules with the finite exchange property.

We now want to show that for a module, the finite exchange property is equivalent to the 2-exchange property.

Theorem 1.26. *A module has the finite exchange property if and only if it has the* 2*-exchange property.*

Proof. We can prove this result by induction, in particular we want to show that if M has the n-exchange property than it has the $n+1$ exchange property for all $n \geq 2$. Let now $G = M' \oplus N = \oplus_{i=1}^{n+1} A_i$ and $H = \oplus_{i=1}^{n} A_i$. Since M has the n-exchange property it has also the 2-exchange property, so there exist $A'_{n+1} \leq A_{n+1}$ and $H' \leq H$ such that $G = M' \oplus A'_{n+1} \oplus H'$. By the usual techniques, we get $H = H' \oplus H''$, where $H'' = H \cap (M' \oplus A'_{n+1})$, and $A_{n+1} = A'_{n+1} \oplus B_{n+1}$, where $B_{n+1} = A_{n+1} \cap (H' \oplus M')$. From

$$G = M' \oplus H' \oplus A'_{n+1} = (H'' \oplus B_{n+1}) \oplus (H' \oplus A'_{n+1})$$

, we notice that H'' must be isomorphic to a direct summand of M' and so has the n-exchange property (Theorem 1.25). Therefore there exist $B_i \subseteq A_i$ such that $H = H'' \oplus (\oplus_{i=1}^{n} B_i)$. Since $H'' \leq M' \oplus A'_{n+1} \subseteq H'' \oplus H' \oplus A_{n+1}$ we have a decomposition $M' \oplus A'_{n+1} = H'' \oplus H'''$ in the usual way. Then

$$G = H''' \oplus H = B_1 \oplus \cdots \oplus B_n \oplus A'_{n+1} \oplus M'.$$

□

Now, we want to to show that every quasi-injective module has the exchange property.

Theorem 1.27. *Let M be a quasi-injective module. Then M has the exchange property.*

Proof. By Theorem 1.21, every quasi-injective module is continuous. Let $G = M \oplus C = \oplus_{i \in I} N_i$. Set $N'_i = N_i \cap C$. Consider now the set Ω consisting of the modules P such that:

$P = \oplus_{i \in I} P_i$, with $N'_i \leq P \leq N_i$ and $P \cap M = 0$. Ordering Ω by set inclusion it is easy to see that we can apply Zorn's Lemma, so we can find a maximal element $X = \oplus_{i \in I} X_i$ of Ω. Consider now Q_j, $j \in J$, such that $X_j < Q_j \leq N_j$, and call h the projection of G on G/X. By the maximality of X, it follows that $(X + Q_j) \cap M \neq 0$ and $(Q_j + X) \cap M$ is not contained in X. This implies that

$$0 \neq h(M) \cap h(Q_j + X) = h(M) \cap h(Q_J) = h(M) \cap h(N_j) \cap h(Q_j).$$

This means in particular that $h(M) \cap h(N_j)$ is essential in $h(N_j)$. Hence, $\oplus_{j \in I}(h(M) \cap h(N_j))$ is essential in $\oplus_{i \in I} h(N_i) = h(G)$. In particular, $h(M)$ is essential in G.

Consider now the projection π_C of G over C. Since $N_i \cap Ker(\pi_C) = N_i'$ it follows that N_i/N_i' is isomorphic to a submodule of M. Let $N' = \oplus_{i\in I} N_i'$. By Theorem 1.10, we obtain that M, being M-injective, is $G/N' = \oplus_{i\in I} N_i/N_i'$-injective. Since $h(G)$ is an homomorphic image of G/N' we get that M is $h(G)$-injective and therefore $h(M) \cong M$ is $h(G)$-injective. From Theorem 1.13, we get that $h(M)$ is a direct summand of $h(G)$, and since $h(M)$ is essential in $h(G)$, we get that $G = M \oplus X$.

□

We say that a ring R has the *exchange property* if, for any two elements r_1 and r_2 of R such that $r_1 + r_2 = 1$, there exist two idempotents $e \in r_1R$, $f \in r_2R$ such that $e + f = 1$.

Theorem 1.28. *Let M be a right S-module and let $R = End(M_S)$. Then the following are equivalent:*

(a) The module M has the finite exchange property.

(b) The ring R is exchange.

(c) For any two elements r_1 and r_2 of R such that $r_1 + r_2 = 1$ there exist two idempotents $e \in Rr_1$, $f \in Rr_2$ such that $e + f = 1$.

The previous theorem says in particular that the definition we have given on the right for an exchange ring, is equivalent to the one on the left given in Theorem 1.28 (c). The next result gives consistence to the nomenclature used:

Theorem 1.29. *The following are equivalent for a ring R:*

(a) R is an exchange ring.

(b) The module R_R is exchange (it is equivalent to ask the same condition on the left).

(c) If $f_1 + \cdots + f_n = 1$, with the f_i idempotents, then there exist orthogonal idempotents e_i, for $i = 1, \cdots, n$, such that $e_1 + \cdots + e_n = 1$ and $e_i \in Rf_i$ for every i (this condition is left-right symmetric).

(d) Every finitely generated projective right R-module is an exchange module (this condition is left-right symmetric).

(e) There is a unitary subring S of R such that S is exchange.

(f) The ring R is isomorphic to a factor ring of a product of exchange rings.

Theorem 1.30. *The following are equivalent for a non-zero right R-module M:*

(a) The module M has the finite exchange property.

(b) There exists a direct sum decomposition $M = \oplus_{i=1}^{n} M_i$ *such that all the endomorphism rings* $End(M_i)$ *are exchange rings.*

(c) There exist orthogonal idempotents $e_1, \ldots, e_n \in End(M)$ *such that*

$$e_1 + \cdots + e_n = 1$$

and $e_i End(M) e_i$ *is an exchange ring for every* $i = 1, \ldots, n$.

Let R be a ring and I a two sided ideal of R. We say that *the idempotents of R lift modulo I* if, every time we have an idempotent element $x+I$ in R/I, there exists an idempotent $e \in R$ such that $e - x \in I$. The next theorem points out that a ring R is exchange if and only if $R/J(R)$ is exchange and the idempotents of R lift modulo $J(R)$. This result is of particular importance and will be used to derive important results in the next chapters.

Theorem 1.31. *For a ring R the following are equivalent:*

(a) The ring R is exchange.

(b) There exists an ideal $I \subseteq J(R)$ *such that* R/I *is an exchange ring and idempotents can be lifted modulo I.*

(c) The ring $R/J(R)$ *is exchange and idempotents of R lift modulo* $J(R)$.

(d) For every element $x \in R$ *there exists an idempotent* $e \in R$ *such that* $R = eR + (1 - x)R$.

(e) every idempotent of R can be lifted modulo every right ideal (this property is left-right symmetric).

It is not difficult to show that a ring R is an exchange ring without nontrivial idempotents if and only if R is local. It can be also shown that:

Theorem 1.32. *A module M is an indecomposable exchange module if and only if End(M) is local.*

In particular, if we have an exchange module M which is a finite direct sum of idecomposables, all its direct summands have local endomorphism ring. It is then natural the following theorem:

Theorem 1.33. *The following are equivalent for a non-zero module M :*

(a) The module M is exchange and is the direct sum of finitely many indecomposable modules (or M is a finite direct sum of modules with local endomorphism ring).

(b) The module M is a finite direct sum of modules whose endomorphism rings are semiperfect.

(c) The ring End(M) is semiperfect.

Chapter 2

Modules invariant under automorphisms of their envelopes

2.1 First properties of envelopes

We know that every right R-module M has an injective envelope $E(M)$ with the nice properties we recalled in the previous chapter. We would like now to take advatage of those properties, in order to give a more general definition of "envelope".

Given a ring R and a class χ of right R-modules closed under isomorphisms, a χ-*preenvelope* of a right R-module M, is a morphism $u : M \to X$, with $X \in \chi$, such that, for every morphism $u' : M \to X'$, with $X' \in \chi$, there exists an homomorphism $f : X \to X'$ that factors u', in other words, $u' = f \circ u$.

The definition of a χ-preenvelope is not completely satisfactory. In fact, assume that M is a module with a χ-preenvelope $u : M \to X$. To build a new concept of "envelope", we would like to have unicity up to isomorphism of the module X, analogously to the case of injective envelopes. It is then natural to give the following definition: we say that a χ-preenvelope $u : M \to X(M)$ is a χ-envelope if for every morphism of right R-modules $h : X(M) \to X(M)$ such that $h \circ u = u$, h is an automorphism.

It is easy to prove the following:

Theorem 2.1. *Suppose that M is a module with χ-envelope $u : M \to X(M)$. Then, if $u' : M \to X$ is an another χ-envelope of M, X is isomorphic to $X(M)$.*

Proof. Suppose that $u : M \to X(M)$ and $u' : M \to X'$ are two envelopes of M. By the definition of a χ-preenvelope, we know that there exist two morphisms $f : X(M) \to X'$ and $f' : X' \to X(M)$ such that $u' = f \circ u$, $u = f' \circ u'$. We immediately obtain that $f \circ f' \circ u' = u'$ and $f' \circ f \circ u = u$. By the definition of χ-envelope, we obtain that $f \circ f'$ and $f' \circ f$ are isomorphisms, so f and f' are isomorphisms too. □

We say that the class χ is *enveloping*, if each right R-module has a χ-envelope. In particular, the class of injective objects is an enveloping class.

When a χ-(pre)envelope u of a module M is monomorphic we say that the χ-(pre)envelope is *monomorphic*. As in the injective case, it is natural to ask whether a direct sum decomposition of a module M of type $M = M_1 \oplus M_2$, where M_1 and M_2 are right R-modules with a χ-envelope, leads to a χ-envelope of M. By the following theorem, the answer is positive.

Theorem 2.2. *Let M be a right R-module and let M_1 and M_2 be submodules of M such that $M = M_1 \oplus M_2$. Suppose that M_1 and M_2 have χ-(pre)envelopes, respectively $u_1 : M_1 \to X(M_1)$ and $u_2 : M_2 \to X(M_2)$. Then $u_1 \oplus u_2 : M \to X(M_1) \oplus X(M_2)$ is a χ-(pre)envelope of M.*

Proof. If $u' : M \to X'$ is a morphism, with $X' \in \chi$, then we have that $u' \circ \iota_{M_1} = f_1 \circ u_1$ and $u' \circ \iota_{M_2} = f_2 \circ u_2$ for some $f_1 : X(M_1) \to X'$, $f_2 : X(M_2) \to X'$ (here ι_{M_1} and ι_{M_2} are just the inclusions of M_1 and M_2 into M). By the Universal Property of Direct Sums, we get that there exists $f : X(M_1) \oplus X(M_2) \to X'$ such that $f \circ \iota_{X(M_1)} = f$ and $f \circ \iota_{X(M_2)} = f_2$ (here $\iota_{X(M_1)}$ and $\iota_{X(M_2)}$ are just the inclusions of $X(M_1)$ and $X(M_2)$ into $X(M_1) \oplus X(M_2)$). It is immediate to verify that $f \circ (u_1 \oplus u_2) = u'$, so in particular $u_1 \oplus u_2$ is a χ-preenvelope of M. It is easy to verify that $u_1 \oplus u_2$ is also a χ-envelope of M. □

2.2 Automorphism and endomorphism invariant modules

In this section, given a module M, we will usually use the expression "χ-envelope of M" referring to an object of the class of isomorphic objects $\{X \in \chi \mid \exists u : M \to X \ \chi\text{-envelope}\}$ or to couples of the form $(u, X(M))$, where $u : M \to X(M)$ is a χ-envelope of M.

As in the injective case, we could ask which modules are invariant under endomorphisms or automorphisms of their χ-envelope.

We say that a module M, with χ-envelope $X(M)$, is χ-*endomorphism invariant* if, for every endomorphism $g : X(M) \to X(M)$, there exists an endomorphism f of M such that $g \circ u = u \circ f$.

We say that a module M, with χ-envelope $X(M)$, is χ-*automorphism invariant* if, for every automorphism $g : X(M) \to X(M)$, there exists an endomorphism f of M such that $g \circ u = u \circ f$.

It is pretty natural to wonder whether the endomorphism f in the last definition is an automorphism. The answer is positive whenever the χ-envelope u is monomorphic.

Theorem 2.3. *Suppose that M has a monomorphic envelope $u : M \to X(M)$ and that M is χ-automorphism invariant. Then, if g is an automorphism of $X(M)$, and f is an endomorphism of M such that $g \circ u = u \circ f$, f is an isomorphism.*

Proof. Since g is an automorphism there exists an endomorphism f' of M such that $g^{-1} \circ u = u \circ f'$. Then $u = u \circ f' \circ f = u \circ f' \circ f$, and since u is monomorphic we have that f is an isomorphism. □

It is worth noticing that similarly to the case of a quasi-injective module M, where we built a ring homomorphism $\alpha : \mathrm{End}(E(M)) \to \mathrm{End}(M)$ (Theorem 1.18), we can build a group homomomorphism $\Delta : Aut(X(M)) \to Aut(M)$ for every χ-automorphism invariant module M with a monomorphic χ-envelope $u : M \to X(M)$, associating to an automorphism g the automorphism f of M such that $g \circ u = u \circ f$ (it is well defined since u is a monomorphism). In particular, the kernel of the map Δ, is given by the automorphisms g such that $g \circ u = u$. We will call this kernel the *Galois group* of the envelope u and we indicate it with the symbol $\mathrm{Gal}(u)$.

We will see in the next sections, examples of χ-automorphism invariant modules that are not χ-endomorphism invariant. But we can prove in some cases that a χ-automorphism invariant module is χ-endomorphism invariant:

Theorem 2.4. *Suppose that M is a χ-automorphism invariant right R-module with χ-envelope $M \xrightarrow{u} X(M)$ and suppose that every element of*

$$End(X(M)_R)/J(End(X(M)_R))$$

is the sum of units, then M is χ-endomorphism invariant.

Proof. Suppose we can show that every element of $\mathrm{End}(X(M))$ can be written as the sum of some units. Then since M is χ-automorphism invariant we obtain immediately that M is χ-endomorphism invariant.

We now have to show that the fact that each element of

$$\mathrm{End}(X(M)_R)/J(\mathrm{End}(X(M)_R))$$

can be written as the sum of two units implies that every element of

$$Q = \mathrm{End}(X(M))$$

is the sum of two units. Suppose that $r + J(Q) = i + j + J(Q)$ where r, i, $j \in R$ and $i + J(Q)$ and $j + J(Q)$ are units. Since $i + J(Q)$ is a unit there exists an $i' \in R$ such that $1 - ii' \in J(Q)$. By the usual properties of the Jacobson radical, we have that $1 - (1 - ii') = ii'$ is invertible in R. We can do the same on the other side and in particular we obtain that i is invertible (the same thing holds for j). By construction, there exists an $r' \in J(Q)$ such that $r = i + j + r'$, but then it is sufficient to rewrite this equality as $r = i + j + (r' - 1) + 1$ to conclude. □

Theorem 2.4 suggests that to study χ-endomorphism invariant modules, it is important to study the unit structure of rings.

We now give a result which answers the following question: is a direct summand of a χ-automorphism invariant module, χ-automorphism invariant?

Theorem 2.5. *Let M be a χ-automorphism invariant module and assume that every direct summand N of M has a χ-envelope.*

Then every direct summand of M is χ-automorphism invariant.

2.3 Von Neumann regular rings

Von Neumann regular rings will play a fundamental role in the next sections, so we will collect here some results.

We say that a ring R is *Von Neumann regular* if for every element $a \in R$ there exists an element $x \in R$ such that $a = axa$. It is not difficult to prove the following:

Theorem 2.6. *A ring is Von Neumann regular if and only if every finitely generated right ideal is generated by an idempotent.*

Proof. Suppose first that every finitely generated right ideal can be generated by an idempotent. Let $a \in R$, then $aR = eR$ for an idempotent $e \in R$. This implies that there exist $x, x' \in R$ such that $ax = e, ex' = a$, but then $axa = e^2x' = ex' = a$. Now suppose that R is Von Neumann regular. It is easy to show that every cyclic right ideal $I = aR$ is generated by an idempotent. Consider in fact $x \in R$ such that $axa = a$. Then $axR = aR$. To show that every finitely generated right ideal I is generated by an idempotent it suffices to show that if I is generated by two elements, then I is generated by ad idempotent. Suppose $I = x_1R + x_2R$, then there exists $e_1 \in R$ such that $x_1R = e_1R$ and we have that $I = e_1R + (1 - e_1)x_2R$. Consider now an idempotent $e_2 \in R$ such that $(1 - e_1)x_2R = e_2R$ and set $e_3 = e_2(1 - e_1)$. In particular there is an element $x_3 \in R$ such that $e_2 = (1 - e_1)x_2x_3$. It is then easy to notice that $e_3e_1 = 0$, $e_1, e_3 = 0$, $e_2R = e_3R$ and that e_3 is an idempotent. Thus $I = e_1R+e_3R$, e_1+e_3 is an idempotent and $I = (e_1+e_3)R$; in fact for every element e_ir_i in e_iR we want to build it is sufficient to consider $(e_1 + e_3)e_ir_i = e_ir_i$. □

It is an immediate consequence of Theorem 2.6 that the Jacobson radical of a Von Neumann regular ring R is zero; in fact if $x \in J(R)$, then $xR = eR \subseteq J(R)$ for some idempotent $e \in R$, but as $(1 - e)$ is invertible, we get that $e = 0$.

We can now prove the following:

Theorem 2.7. *Every Von Neumann regular ring has the exchange property.*

Proof. Theorem 2.6 implies that a Von Neumann regular ring S satisfies the property that if r_1 and r_2 are two elements of S such that $r_1 + r_2 = 1$, there exist $e \in r_1R$ and $f \in r_2R$ idempotents such that $f + e = 1$. □

Another important characterization of Von Neumann regular rings is the following:

Theorem 2.8. *A ring R is Von Neumann regular if and only if every right R-module is flat.*

Let R be a ring and let M be a right R-module. We indicate by $Z(M)$ the set of all $m \in M$ such that $r(m) \leq_e R$ and we call it the *singular submodule* of M (it is not difficult to show that $Z(M)$ is a submodule of M). We say that a module M is *singular* if $Z(M) = M$ and we say that M is *nonsingular* if $Z(M) = 0$.

Theorem 2.9. *Let R be a ring. If all the principal right ideals of R are projective, then $Z(R) = 0$.*

Proof. Consider an element $r \in R$ with $r \neq 0$. We have a split exact sequence $0 \to r(r) \to R \to rR \to 0$. Then $r(r)$ is a direct summand of R and since $r(r) \neq R$, we have that $r(r)$ cannot be essential in R. □

Corollary 2.10. *In a Von Neumann regular ring* $Z(R_R) = 0$.

Proof. The proof is straightforward: every principal right ideal in a Von Neumann regular ring is generated by an idempotent, and so it is projective. Hence, applying Theorem 2.9 we can conclude. □

Corollary 2.11. *In a Von Neumann regular ring the right annihilator of a principal right ideal is generated by an idempotent element.*

Proof. Look at the proof of Theorem 2.9. □

We state now a theorem that will be useful in the last chapter.

Theorem 2.12. *A commutative ring is Von Neumann regular if and only if the simple right R-modules are injective.*

Now we give some results about singular and nonsingular modules that will be helpful in the future.

Theorem 2.13. *Let* M *be a right* R*-module. Then* M *is nonsingular if and only if* $Hom_R(N, M) = 0$ *for every singular right* R*-module* N.

Proof. If M is nonsingular, then for very morphism $f : N \to M$, where N is a singular submodule, we have that $f(N) = f(Z(N)) \leq Z(M) = 0$. Then $Hom_R(N, M) = 0$.

Let's now show the opposite. We have that $Hom(Z(M), M) = 0$, so also the natural inclusion of $Z(M)$ into M is 0 and we can conclude. □

Theorem 2.14. *Let* M*,* N *and* C *be right* R*-modules such that* $N \leq M$*,* M/N *is singular and* C *is nonsingular. Then if* $g_i : M \to C$ *for* $i = 1, 2$ *are two homomorphisms that coincide on* N*, we have that* $g_1 = g_2$.

Proof. Since by Theorem 2.13 $Hom_R(M/N, C) = 0$, we have an exact sequence $0 \to Hom_R(M, C) \to Hom_R(N, C)$. □

2.4 Endomorphism rings of quasi-injective modules

Consider a quasi-injective module M. In this section, we want to prove that the $\mathrm{End}(M)/J(\mathrm{End}(M))$ is Von Neumann regular, right self-injective and that $\mathrm{End}(M)$ has the property of the lifting of the idempotents modulo the Jacobson radical.

Theorem 2.15. *Let M be a quasi-injective right R-module and let Q=End(M_R) be its endomorphism ring. Then $J(Q) = \{f \mid ker(f) \leq_e M\}$ and $Q/J(Q)$ is Von Neumann regular.*

Proof. Let's show that $K = \{f \mid ker(f) \leq_e M\}$ is a two sided ideal of Q. If f, $g \in K$, then $Ker(f) \cap Ker(g) \subseteq Ker(f+g)$ and so $ker(f+g) \leq_e M$. Consider now any element $h \in Q$. Since $ker(f) \subseteq ker(h \circ f)$ then $Ker(h \circ f) \leq_e M$ and since $h^{-1}(Ker(f)) \subseteq Ker(f \circ h)$ and $h^{-1}(Ker(f)) \leq_e M$ we have that $Ker(f \circ h) \leq_e M$. So we have obtained that both $h \circ f$ and $f \circ h$ belong to K, and this means that K is a two sided ideal of R.

We want to show that $K \subseteq J(Q)$; we know that the Jacobson radical of a ring S coincides with the elements $r \in S$ such that $1 - sr$ is left invertible for very $s \in S$, but since K is an ideal it is sufficient to show that for any element $f \in K$ we have that $(1-f)$ is left invertible. Since $ker(f) \cap ker(1-f) = 0$ we have that $(1-f)$ is a monomorphism. So we can consider $h : (1-f)(M) \to M$, the inverse map of $1-f$ on $(1-f)(M)$. Since M is quasi-injective the map h can be extended to an endomorphism g of M and we have that $g \circ (1-f) = 1$.

Since the Jacobson radical of a Von Neumann regular ring is 0, in order to show that $K = J(Q)$ it is enough to show that Q/K is Von Neumann regular. Let f be an endomorphism of M and let B be a $\cap$-complement of $ker(f)$ in M. Then we have that $f_{|B} : B \to f(B)$ is an isomorphism and we can call h its inverse. We have then that $h \circ f$ is the identity on B and so $(f \circ h \circ f - f)(B) = 0$; this implies that $ker(f) \oplus B \leq Ker(f \circ h \circ f - f)$ and since $Ker(f) \oplus B \leq_e M$ we have that $f \circ h \circ f - f \in K$, hence, $Q/J(Q)$ is Von Neumann regular.

□

By the following theorem, if M is a quasi-injective right R-module, the surjective ring homomorphism

$$\alpha : \mathrm{End}(E(M)) \to \mathrm{End}(M),$$

induces an isomorphism

$$\mathrm{End}(M_R)/J(\mathrm{End}(M_R)) \cong \mathrm{End}(E(M_R))/J(\mathrm{End}(E(M_R))).$$

Theorem 2.16. *Consider a quasi-injective right R-module M and define $Q = End(M)$, $T = End(E(M))$. Then the morphism $\alpha : T \to Q$ induces an isomorphism between $T/J(T)$ and $Q/J(Q)$.*

Proof. We know that the morphism α is surjective, so we have only to show that $\alpha^{-1}(J(Q)) = J(T)$. We can obtain easily that $\alpha^{-1}(J(Q)) \leq J(T)$. In fact, if $f \in J(Q)$, then $Ker(f) \leq_e M$ and, if g is an application that extends f, we have that $Ker(f) \leq Ker(g)$; this implies that $Ker(g)$ is essential in $E(M)$ since $Ker(f)$ is essential in M. To show that $\alpha(J(T)) \leq J(Q)$ it is sufficient to consider $g \in J(T)$ and notice that $f = \alpha(g)$ is such that $Ker(g) \cap M \subseteq Ker(f)$, and so $Ker(f)$ is essential in M. □

We already proved that a quasi-injective module has the exchange property, but we will be able to achieve this result in another way.

Theorem 2.7 states that a Von Neumann regular ring is exchange. If we are able to prove that for a quasi-injective module M its endomorphism ring Q has the property of the lifting of idempotents modulo its Jacobson radical, we get that Q is an exchange ring (Theorem 1.28); if this is true, quasi-injective modules satisfy the finite exchange property, therefore the exchange property (we already pointed out that quasi-continuous modules with the finite exchange property satisfy the exchange property; see section 1.8).

The next theorem will show that the endomorphism ring of an injective module has the property of the lifting of idempotents modulo its Jacobson radical. Notice that this automatically implies that the ring of endomorphism of a quasi-injective module has the property of the lifting of idempotents modulo its Jacobson radical. In fact, with the same notations as in Theorem 2.16, we have that every idempotent $\bar{e}$ of $Q/J(Q)$ comes from an idempotent $\bar{f} \in T/J(T)$, and (by the following theorem) $\bar{f}$ lifts to an idempotent f in T. Then $\alpha(f)$ is an idempotent lifting $\bar{e}$.

Theorem 2.17. *Let M be an injective right R-module with endomorphism ring Q. Then every idempotent of Q lifts modulo $J(Q)$.*

Proof. Consider f such that $f - f^2 \in J(Q)$, and call $\bar{f}$ the projection of f in $Q/J(Q)$. Since $\bar{f} - \bar{f}^2 = 0$, $Ker(f - f^2) \leq_e M$.

It is easy to see that $f(Ker(f - f^2)) \leq Ker(1 - f)$ and that

$$(1 - f)(Ker(f - f^2)) \leq Ker(f).$$

We obtain then that $Ker(f - f^2) \subseteq Ker(f) \oplus Ker(1 - f)$, and so

$$Ker(f) \oplus Ker(1 - f)$$

is essential in M. Since M has two direct summands such that their sum is essential, we get that $M = E(Ker(f)) \oplus E(Ker(1 - f))$. Call π_1 and π_2 respectively the projections of M on $E(Ker(f))$ and $E(Ker(1-f))$; we have then that π_2 coincides with f on $Ker(f) \oplus Ker(1 - f)$ and this implies that $\pi_2 - f \in J(Q)$. □

We would like to prove that the endomorphism ring Q of a quasi-injective module M is such that $Q/J(Q)$ is right self-injective, but to do so we need some results.

Lemma 2.18. *Let M be an injective right R-module, let $Q = End(M_R)$ and let $\bar{Q} = Q/J(Q)$ (we will then indicate the projection of an element $f \in Q$ over the factor ring $\bar{Q}$ as $\bar{f}$). Suppose that there exist two idempotents e, $f \in Q$ such that $\bar{e}\bar{Q} \cap \bar{f}\bar{Q} = 0$. Then we have that $e(M) \cap f(M) = 0$.*

Proof. We need to slightly change f in order to obtain $\bar{f}\bar{e} = 0$. Since $\bar{Q}$ is Von Neumann regular by Theorem 2.15, we have that $\bar{e}\bar{Q} \oplus \bar{f}\bar{Q}$ is generated by an idempotent $\bar{q}$, hence, there exists a right ideal $I \in \bar{Q}$ such that $\bar{Q}_{\bar{Q}} = \bar{e}\bar{Q} \oplus \bar{f}\bar{Q} \oplus I$. Since $J = \bar{e}\bar{Q} \oplus I$ and $\bar{f}\bar{Q}$ are such that $J \oplus \bar{f}\bar{Q} = \bar{Q}_{\bar{Q}}$, there exists an idempotent $x \in \bar{Q}$ such that $x\bar{Q} = \bar{f}\bar{Q}$ and $(1-x)\bar{Q} = J$. By Theorem 2.17, there exists an idempotent $h \in Q$ such that $\bar{h} = x$, and in particular we have that $\bar{h}\bar{Q} = \bar{f}\bar{Q}$ and $\bar{h}\bar{e} = 0$ (this is true since there exists a $q \in \bar{Q}$ such that $\bar{e} = (1-x)q$). Since $\bar{f}\bar{Q} = \bar{h}\bar{Q}$ we have that $\bar{f}\bar{h} = \bar{h}$ and $\bar{h}\bar{f} = \bar{f}$, in fact there exists a $q'' \in \bar{Q}$ such that $\bar{h}q'' = \bar{f} = \bar{h}\bar{h}q'' = \bar{h}\bar{f}$. We have then that $\bar{f} + \bar{f}\bar{h}(1-\bar{f}) = \bar{h}$ and we set $g = f + fh(1-f)$. Clearly g is an idempotent and $\bar{g}\bar{Q} = \bar{f}\bar{Q}$. Since $fg = g$ and $gf = f$ we immediately get that $f(M) = g(M)$.

We now substitute f with g so that we can suppose that $\bar{f}\bar{Q} \cap \bar{e}\bar{Q} = 0$ and $\bar{f}\bar{e} = 0$. Since $fe \in J(Q)$ we have that $K = Ker(fe) \leq_e M$, and this implies that $K \cap eM \leq_e e(M)$. It can be easily verified that $e(K) = K \cap e(M)$. Therefore $e(K) \leq_e eM$ and since $fe(K) = 0$ we have that $f(M) \cap e(K) = 0$ (if $f(M) \cap e(K) \neq 0$ there exist elements $m \in M$, $k \in K$ such that

$$f(m) = e(k) = f^2(m) = f(e(k)) = 0).$$

Since $e(K)$ is essential in $e(M)$ we instantly get that $e(M) \cap f(M) = 0$. □

Lemma 2.19. *Let M be an injective right R-module and let $Q = End(M_R)$. Then if e_α is a collection of idempotents of Q such that $e_\alpha\bar{Q}$ is an independent family of right ideals of $\bar{Q}$, we get that $e_\alpha(M)$ is an independent family of submodules of M.*

Theorem 2.20. *Let M be a quasi-injective right R-module and call Q its endomorphism ring. Then $Q/J(Q)$ is right self-injective.*

Proof. By Theorem 2.16, we can suppose M to be injective. To show that $\bar{Q}$ is right-self injective, it is sufficient to show that for every morphism f from a right ideal I of $\bar{Q}$ to $\bar{Q}$, there exists an element $w \in \bar{Q}$ such that f is just the multiplication by w (remember $\mathrm{End}(\bar{Q}_{\bar{Q}}) \cong \bar{Q}$).

Consider now a maximal family C_α of independent cyclic submodules of I. We know that $\oplus_\alpha C_\alpha \leq_e I$ and since $\bar{Q}$ is regular by Theorem 2.15 every C_α is generated by an idempotent of $\bar{Q}$ and by Theorem 2.15 these idempotents lift to idempotents $e_\alpha \in Q$. Then $e_\alpha \bar{Q}$ is a family of independent cyclic submodules of $\bar{Q}$ generated by idempotents of Q, and by Lemma 2.19 $e_\alpha(M)$ is an idependent family of submodules of M. For each element e_α consider an element $t_\alpha \in \bar{Q}$ such that $\bar{t_\alpha} = f(\bar{e_\alpha})$ and consider the restriction of the maps t_α to $e_\alpha(M)$. This family of maps, by the Universal property of direct sums, defines uniquely a morphism $g : \oplus_\alpha e_\alpha(M) \to M$, and by the injectivity of M this map extends to an endomorphism w of M such that $w \circ e_\alpha = t_\alpha \circ e_\alpha$. Now $\bar{w} \circ \bar{e}_\alpha = \bar{t}_\alpha \circ \bar{e}_\alpha = f(\bar{e_\alpha}) \circ \bar{e_\alpha}$, so f agrees with the left multiplication by $\bar{w}$ in $\oplus_\alpha e_\alpha \bar{Q}$. We also have that $I/\oplus_\alpha e_\alpha \bar{Q}$ is singular, in fact for every $\bar{i} \in I/\oplus_\alpha e_\alpha \bar{Q}$, $r(\bar{i}) \neq 0$, but if $r(\bar{i})$ is not essential in $\bar{Q}_{\bar{Q}}$ then there exists a right ideal $J \neq 0$ in $\bar{Q}$ such that $J \cap r(\bar{i}) = 0$ and, calling i a preimage of $\bar{i}$, we get that iJ is a non-zero submodule of I that has zero intersection with $\oplus_\alpha e_\alpha \bar{Q}$, absurd since $\oplus_\alpha e_\alpha \bar{Q}$ is essential in I. Since by Theorem 2.9 $Z(\bar{Q}) = 0$ and since $I/\oplus_\alpha e_\alpha \bar{Q}$ is singular we have, by Theorem 2.14, that left multiplication by w agrees with f on I.

□

2.5 Pure-injective envelopes

In this section, we will informally discuss the existence of *pure-injective* envelopes for right R-modules and we will see that the endomorphism ring of a *pure-injective* module has some nice properties. In order to do so, we have to recall few definitions from category theory.

We say that a category **C** is *Grothendieck* if:

(a) The category **C** is abelian.

(b) The category **C** has arbitrary coproducts.

(c) The category **C** has a generator. This means that there exists an object G in **C** such that the functor $Hom(G, -) : \mathbf{C} \to \mathbf{Sets}$ is faithful.

(d) Direct limits are exact in **C**.

Clearly the category R-Mod is an example of a Grothendieck category.

We denote with the symbol $_{\mathbf{R}}\mathbf{FP}$ the full subcategory of R-Mod whose objects are the finitely presented left R-modules, and we denote by **Ab** the category of abelian groups. We can consider now the category $(_{\mathbf{R}}\mathbf{FP}, \mathbf{Ab})$

whose objects are the additive functors from the category ${}_{\mathbf{R}}\mathbf{FP}$ to the category of abelian groups, and whose morphisms are the natural transformation between functors. It can be proved that $({}_{\mathbf{R}}\mathbf{FP}, \mathbf{Ab})$ is a Grothendieck category and it is known that in a Grothendieck category every object has an injective hull.

We would like now to include in some sense the cateogry of right R-modules into the category $({}_{\mathbf{R}}\mathbf{FP}, \mathbf{Ab})$ in order to have more tools to study it. It is natural in fact to consider, for every right R-module M, the functor $M \otimes_R - : {}_{\mathbf{R}}\mathbf{FP} \to \mathbf{Ab}$, and it can be proved that the functor

$$\text{Mod-}R \to ({}_{\mathbf{R}}\mathbf{FP}, \mathbf{Ab})$$

defined by $M \to M \otimes_R -$ is fully faithful (it will be clear in a moment why we are considering this functor).

Given two right R-modules M, N we say that a monomorphism $f : M \to N$ is *pure* (or we say that M is a *pure submodule* of N) if for every left R-module X the application $f \otimes id : M \otimes X \to N \otimes_R X$ is a monomorphism. We say that a module M is *pure-injective* if, for every module N, every morphism from a pure submodule N' of N to M can be extended to a morphism from N to M. More precisely, M is pure-injective if every time we have a pure monomorphism $f : N' \to N$ between two modules and a morphism $g : N' \to M$ we can find a morphism $h : N \to M$ such that $h \circ f = g$. One could verify that a sequence of right R-modules $0 \to A \xrightarrow{f} B \to C \to 0$ is pure exact (i.e. short exact with f pure a monomorphism) if and only if the induced sequence of functors $0 \to A \otimes_R - \to B \otimes_R - \to C \otimes_R - \to 0$ is exact in $({}_{\mathbf{R}}\mathbf{FP}, \mathbf{Ab})$.

We will not give a proof of the following theorem which represents the key result in this section:

Theorem 2.21. *An object $F \in ({}_{\mathbf{R}}\mathbf{FP}, \mathbf{Ab})$ is injective if and only if it is isomorphic to a functor of the type $M \otimes_R -$, where M is a pure-injective right R-module. Hence, the full subcategory of Mod-R whose objects are the pure-injective right R-modules, is equivalent to the full subcategory of $({}_{\mathbf{R}}\mathbf{FP}, \mathbf{Ab})$ whose objects are the injective objects of $({}_{\mathbf{R}}\mathbf{FP}, \mathbf{Ab})$.*

Given a module M, we will say that M has a *pure-injective envelope* if there exists a pure-injective right R-module $PE(M)$ and a pure monomorphism $i : M \to PE(M)$ that does not factor through any direct summand of $PE(M)$. It can be proved that:

Theorem 2.22. *Every right R-module M has a pure-injective envelope.*

Considering now the class χ of pure-injective right R-modules, it can be shown that a pure-injective envelope is a χ-envelope.

But we can say more:

Theorem 2.23. *For each injective object E in a Grothendieck category $\mathbf{C}$ the factor ring $End_{\mathbf{C}}(E)/J(End_{\mathbf{C}}(E))$ is Von Neumann regular, right self-injective and idempotents lift modulo the Jacobson radical.*

A direct consequence is that:

Theorem 2.24. *Let M be pure-injective right R-module.*
Then $End(M_R)/J(End(M_R))$ is Von Neumann regular, right self-injective and the idempotents of $End(M_R)$ lift modulo the Jacobson radical.

Theorem 2.24 and Theorem 2.20 show the importance of studying Von Neumann right self-injective rings and the property of the lifting of idempotents in order to gain more results about injective and pure-injective envelopes. In the next section we will connect this problem to Theorem 2.5 and so we will study the unit structure in a Von Neumann regular right self-injective ring.

2.5.1 Examples

Example 2.5.1. Clearly every injective module is pure-injective. The *Prüfer group* $\mathbb{Z}(p^\infty)$ is a pure-injective $\mathbb{Z}$-module.

Theorem 2.25. *Let M be an artinian right R-module and let $S = End(M_R)$. Then ${}_SM$ is pure-injective.*

Example 2.5.2. Every abelian group of type $\mathbb{Z}/p^n\mathbb{Z}$ is pure-injective (it is both artinian and noetherian) but not injective (the injective hull is $\mathbb{Z}(p^\infty)$).

Example 2.5.3. If we consider a Von Neumann regular ring R, every monomorphism between two modules is pure. In particuar, a right R-module is pure-injective if and only if it is injective. The module $\mathbb{Q}_{\mathbb{Q}}$ is then an example of a pure-injective module.

Theorem 2.26. *Let R be a ring. Every direct summand of a topological compact Hausdorff R-module is pure-injective.*

Example 2.5.4. If R is a commutative local noetherian ring, with maximal ideal $\mathfrak{m}$, its $\mathfrak{m}$-adic completion is the pure-injective envelope of R seen as an R-module.

Example 2.5.5. Let k be a field. Then the power series ring $k[[X]]$ is pure-injective as a k-module.

Example 2.5.6. The ring of p-adic integers $\mathbb{Z}_p$ is pure-injective as a $\mathbb{Z}$-module.

2.6 Von Neumann regular right self-injective rings

In the beginning of this section, we will collect a few results about Type theory of Von Neumann regular right self-injective rings. Then we will procede to analize the unit structure in such rings.

We say that a right R-module A is *directly finite* if A is not isomorphic to any of its direct summands. More formally, we say that A is directly finite if every time we have $A \cong A \oplus B$ for some right R-module B, then B=0. If a module is not directly finite we say that it is *directly infinite*.

It is obvious that every module with finite Goldie dimension is directly finite.

We can prove the following:

Lemma 2.27. *A right R-module A is directly finite if and only if for every $x, y \in End_R(A)$ such that $xy = 1$, we have that $yx = 1$.*

Proof. Suppose that for every $x, y \in End_R(A)$ such that $xy = 1$, $yx = 1$. If A is not directly finite then $A = B \oplus C$ with $B \cong A$ through some isomorphism y_1 and consider $y = \iota_B \circ y_1$. Define now the unique module endomorphism $x : A \to A$ such that $x_{|B} = y_1^{-1}$ and $x_{|C} = 0$. We then obtain that $xy = 1$ but $yx \neq 1$.

Now we want to show that if there exist two endomorphisms x, y of A such that $xy = 1$ and $yx \neq 1$ then A is not directly finite. Notice that yx is idempotent and $yxy = y$. We have now a decomposition of the type $A = yx(A) \oplus (1 - xy)(A)$ and $y(A) \subseteq yx(A)$ (in fact $yxy(a) = y(a)$). This implies that $y(A) = yx(A)$ and it remains to show that y is a monomorphism, but this is straightforward since $y(a) = 0$ implies that $xy(a) = 1(a) = 0$. □

Lemma 2.27 suggests the following definition:

We say that a ring R is *directly finite* if $xy = 1$ implies $yx = 1$ for every $x, y \in R$.

We say that an idempotent e in a ring R is *abelian* if the ring eRe is abelian (this means that the idempotents of eRe are central in eRe). We say that an idempotent e of a ring R is *directly finite* if eRe is directly finite as a ring (or equivalently if eR is directly finite as a right R-module). Now suppose that the ring R we are considering is Von Neumann regular right self-injective. Then we say that an idempotent $e \in R$ is *faithful* if 0 is the only central idempotent of R orthogonal to e (using Theorem 2.11 it is easy to show that it is equivalent to ask eR to be a right faithful module).

A Von Neumann regular right self-injective ring is said to be *purely infinite* if it contains no non-zero directly finite central idempotents.

We have to fix some nomenclature.

Suppose that R is a Von Neumann regular right self-injective ring:

(a) The ring R is of Type I if R has a faithful abelian idempotent (it can be shown that an abelian idempotent is directly finite).

The ring R is of Type I_f if R is of Type I and directly finite.

The ring R is of Type I_∞ if R is of Type I and purely infinite.

(b) The ring R is of Type II if it has a faithful directly finite idempotent but no non-zero abelian idempotents.

The ring R is of Type II_f if it is of Type II and R is directly finite.

The ring R is of Type II_∞ if it is of type II and R is purely infinite.

(c) The ring R is of Type III if it contains no non-zero directly finite idempotents.

The following theorem gives a charcaterization of Von Neuman regular right self-injective rings.

Theorem 2.28. *A Von Neumann regular right self-injective ring is uniquely (up to isomorphism) a direct product of rings of types $I_f, I_\infty, II_f, II_\infty, III$.*

We will use Theorem 2.28 to give other decompositions of Von Neumann regular right self-injective rings, but before giving them, we need to introduce some other definitons and results.

We say that a matrix $M = (r_{i,j})$ over a ring R is *diagonal* if $r_{i,j} = 0$ for every $i \neq j$. A ring R is said to be an *elementary divisor ring* if every matrix over R is equivalent to a diagonal matrix.

Lemma 2.29. *Let R be a Von Neumann regular right self-injective ring which contains no non-zero abelian idempotents. Then there exist idempotents $e_1, e_2, \dots$ such that $n(e_nR) \cong R_R$ (where by $n(e_nR)$ we mean a direct sum of n copies of e_nR).*

Lemma 2.30. *For a Von Neumann regular right self-injective ring R the following are equivalent:*

(a) The ring R is purely infinite.

(b) The module nR_R is isomorphic to a submodule of R_R.

(c) For all positive integers n we have that $nR_R \cong R_R$.

(d) The injective envelope of $\aleph_0$ *copies of* R_R *is isomorphic to* R_R.

We say that a Von Neumann regular right self-injective ring is of *Type* I_n if there exists a Von Neumann regular abelian ring S such that $R \cong \mathbb{M}_n(S)$.

Lemma 2.31. *A Von Neumann regular right self-injective ring* R *is of type* I_f *if and only if there exist Von Neumann regular right self-injective rings* $R_1, R_2 \dots$ *such that* $R \cong \Pi R_n$ *and each* R_n *is of type* I_n.

2.7 Unit structure of Von Neumann regular right self-injective rings

We now have all the tools to prove the following theorem, which gives a decomposition that will enable us to discuss whether a Von Neumann regular right self-injective ring is such that each of its elements is the sum of two units.

Theorem 2.32. *Let* T *be a Von Neumann regular right self-injective ring. Then* $T \cong T_1 \times T_2$, *where* T_1 *is an abelian Von Neumann regular right self-injective ring and* T_2 *is a Von Neumann regular right self-injective ring in which every element is the sum of two units.*

Proof. Using Theorem 2.28 we can write $T = R_1 \times R_2 \times R_3 \times R_4 \times R_5$ where R_1 is of Type I_f, R_2 is of Type I_∞, R_3 is of Type II_f, R_4 is of Type II_∞ and R_5 is of Type III. Now set $P = R_2 \times R_4 \times R_5$ so that we can write $R = R_1 \times R_3 \times P$, with P purely infinite. From Lemma 2.30, we have that $P_P \cong 2P_P$ and this implies that $P \cong \mathrm{End}(P_P) \cong \mathrm{End}(2P_P) \cong \mathbb{M}_2(P)$. Since regular right self-injective Von Neumann regular rings are elementary divisors rings ([13]), and since product of right self injective Von Neumann regular rings is still right self-injective Von Neumann regular, we have that P is an elementary divisors ring. In $\mathbb{M}_2(P)$ we have then that every element A can be written as the sum of two units. In fact we know that there exists B, Q invertible elements of $\mathbb{M}_2(P)$ such that BAQ is diagonal of type $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$; then $BAQ = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & b \end{bmatrix}$ and so A can be written as the sum of two units. It follows then, since every element of $\mathbb{M}_2(P)$ is sum of two units, then every element of P is the sum of two units.

Since R_3 is of Type II_f, from Lemma 2.29, we have that there exists an idempotent e_2 such that $2(e_2R_3) \cong (R_3)_{R_3}$. This implies that

$$R_3 \cong \mathrm{End}_{R_3}(R_3) \cong \mathrm{End}(2(e_2R_3)) \cong \mathbb{M}_2(e_2R_3e_2).$$

The ring $e_2R_3e_2$ is still Von Neumann regular (for every element $eae \in R$ there exits an $x \in R$ such that $eae = eaexeae = (eae)(exe)(eae))$ and right self-injective, in fact by Theorems 1.11 and 2.31 we get that e_2R is quasi-injective, so $\mathrm{End}(e_2R)/J(\mathrm{End}(e_2R))$ is right self-injective, and since $e_2R_3e_2$ is Von Neumann regular, $J(\mathrm{End}(eR)) = 0$ (remember that $\mathrm{End}(e_2R) \cong e_2Re_2$). As before we get that $e_2R_3e_2$ is an elementary divisor ring, so each element of $\mathbb{M}_2(e_2R_3e_2)$ is the sum of two units, therefore R_3 is a ring in which every element is the sum of two units.

By Lemma 1.12, we get that $R_1 \cong \Pi M_n(S_n)$ where the S_n are abelian Von Neumann regular right self-injective rings, and this implieas that every element in $M_n(S_n)$ is the sum of two units for $n \geq 2$. Writing now $P \times \Pi_{n\geq 2}\mathbb{M}_n(S_2) \times R_3 = T_2$ and $T_1 = S_1$ we can conclude.

□

The next theorem represents a key result that will be applied lots of times. In particular it says that in a right self-injective ring R every element is the sum of two units if and only if the ring has no factor ring isomorphic to $\mathbb{F}_2$.

Theorem 2.33. *For a right self-injective ring R the following are equivalent:*

(a) Every element of R is the sum of two units.

(b) The identity element of R is the sum of two units.

(c) The ring R has no factor ring isomorphic to $\mathbb{F}_2$.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear, and so it suffices to show that $(c) \implies (a)$.

By the Theorem 2.20, $R/J(R)$ is Von Neumann regular and right self-injective, so we can use Theorem 2.32 to obtain a decomposition

$$R/J(R) = R_1 \times R_2$$

where R_1 is Von Neumann regular abelian right self-injective and every element of R_2 is the sum of two units (from the proof of Theorem 2.32, we know that R_1 is right self-injective Von Neumann regular and abelian). Let $S = R_1$. It suffices to show that every element in S is the sum of two units. We proceed by contraddiction.

If there is an element $a \in S$ such that a is not the sum of two units. Let

$$\Omega = \{I \mid I \text{ is a bilateral ideal and } a + I \text{ is not the sum of two units in } S/I\}.$$

The set Ω is non-empty and we can order it by inclusion. If we have a chain $\{I_\lambda \mid \lambda \in \Lambda\}$ we can consider the union $I = \cup I_\lambda$. Suppose that $I \notin \Omega$, so there exist $u, u' \in S$ such that $a + I = u + u' + I$ with $u + I$ and $u' + I$ units; but then there exist an index λ and elements $w, w', i, i' \in S$ such that $uw - 1, u'w' - 1, i, i' \in I_\lambda$ and $a + i = u + u' + i'$. So Ω is inductive and we can apply Zorn's Lemma to obtain a maximal element I. The ring S/I is indecomposable: otherwise $S/I \cong T_1 \times T_2$ and there would be ideals I_1 and I_2 strictly containing I such that $S/I_1 \cong T_1$ and $S/I_2 \cong T_2$, hence, $a + I$ would be a unit in both of them, and so it would be a unit in S/I. This implies that there are no nontrivial central idempotents in S/I (Chinese Remainder Theorem), but since S/I is abelian Von Neumann regular (every finitely generated right ideal is generated by an idempotent), we have that S/I must be an indecomposable division ring. Since $a + I$ is not sum of two units we have that $S/I \cong \mathbb{F}_2$; in fact every element $a \neq 1, 0$ in a division ring D with more than two elements is the sum of two units, for example $a = a + (1 - a)$. □

We say that a ring R is *Boolean* if every element of R is idempotent. Now we show that every right self-injective Von Neumann regular ring can be written as the product of one Boolean ring and a ring in which every element is sum of two units.

Lemma 2.34 (Vamos)**.** *Let M_R be a nonsingular injective module. Then $End_R(M_R)$ is Boolean if and only if the identity morphism of no direct summand of M is a sum of two units.*

Given a ring R we will say that $\mathrm{usn}(R) = n$, where n is a natural number, if every element of R can be written as the sum of exactly n units.

Theorem 2.35. *Let T be an abelian Von Neumann regular right self-injective ring. Then $T = T_1 \times T_2$, where T_1 is such that $T_1 = 0$ or $usn(T_1) = 2$ and T_2 is either zero or a Boolean ring.*

Proof. By Theorem 2.33, it is enough to prove that $T = T_1 \times T_2$ where the unit of T_1 is the sum of two units and T_2 is a Boolean ring. We will prove it using Zorn's Lemma and Lemma 2.34. Let Ω be the set of alla pairs (M, u) where M is a submodule of T_T and u is an automorphism of M such that $id_M - u$ is an automorphism of M. The set Ω is clearly non empty since $(0, 0) \in \Omega$ and we can order it by setting $(M, u) \leq (N, u')$ if $M \leq N$ and $u'_{|M} = u$. One can prove that Ω is inductive and so we can apply Zorn's Lemma and consider a maximal element (T_1, u). Now we want to prove that T_1 is injective. Suppose that $(i, E(T_1))$ is an injective envelope of T_1; by the usual properties of injective envelopes we can extend u to an endomorphism u' of $E(T_1)$. Then, since $i \circ u : T_1 \to E(T_1)$ is an injective envelope of T_1, we can conclude that u' is an isomorphism. It is straightforward to prove that also $id_{E(T_1)} - u'$ is an isomorphism, but then, since (T_1, u) is maximal, it follows that T_1 is injective. Therefore $T = T_1 \oplus T_2$ for some submodule T_2 of T_T. Since T_2 is a direct summand of T_T we have that T_2 is an injective module and it is non-singular since T is Von Neumann regular (Corollary 2.10). Then we can apply Lemma 2.34 to obtain that $\mathrm{End}_T(T_2)$ is a Boolean ring (otherwise we would go against the maximality of (T_1, u)). Since every idempotent in T is central we have that the decomposition $T = T_1 \oplus T_2$ leads to a decomposition $T = T_1 \times T_2$, where the identity element of T_1 is sum of two units by construction and $T_2 \cong \mathrm{End}_T(T_2)$ is a Boolean ring. □

Theorem 2.35 shows that we can give the following decomposition for a right self-injective Von Neumann regular ring:

Corollary 2.36. *Let R be a right self-injective Von Neumann regular ring. Then $R = R_1 \times R_2$, where R_1 is a Boolean ring and every element of R_2 is the sum of two units.*

The following theorem will be use to derive some important properties of χ-automorphism invariant modules.

Theorem 2.37. *Let S be a Von Neumann regular right self-injective ring and R a unitary subring of S which is stable under left multiplication by units of S. Then $R = R_1 \times R_2$ where R_1 is a Booelan ring and R_2 is a right self-injective Von Neumann regular ring in which every element is the sum of two units.*

Proof. By Corollary 2.36, we can write $S = S_1 \times S_2$ where S_1 is a Boolean ring and S_2 is a ring in which every element is the sum of two units. So we can write every element $r \in R$ as $r = s_1 \times s_2$ with $s_1 \in S_1$ and $s_2 \in S_2$. Since R is unitary and since every element $s_2 \in S_2$ is the sum of two units, let's say $s_2 = s_2' + s_2''$ where s_2' and s_2'' are units in S_2, we get that $0 \times s_2 = 1_{S_1} \times s_2' + (-1_{S_1}) \times s_2''$ and in particular this means that $0 \times s_2$ belongs to R for every $s_2 \in S_2$. So it is natural to define $R_2 = S_2$. We have now to look for R_1 and the most natural way to do it is to consider R_1 as the set of the elements $s_1 \in S_1$ such that there exists an $s_2 \in S_2$ with the property that $s_1 \times s_2$ belongs to R. Now if s_1 is an element of R_1 there exists an $s_2 \in S_2$ such that $s_1 \times s_2 \in R$, and since $0 \times s_2 \in R$ we get that $s_1 \times 0 = s_1 \times s_2 - (0 \times s_2)$ is an element in R, so in particular $R = R_1 \times R_2$. Clearly R_2 is Von Neumann regular right self-injective (see the proof of Theorem 2.35), and R_1 is Boolean since S_1 is. □

Corollary 2.38. *Suppose that S is a Von Neumann regular right self-injective ring and let R be a unitary subring of S stable under left multiplication by units of S. If R has no factors isomorphic to $\mathbb{F}_2$ then $R = S$.*

Proof. It is sufficient to show that every element of S is the sum of two units, in fact, since R is stable under left multiplication by elements of S we get that $R = S$. Since S is self-injective we can use Theorem 2.33 to prove our result. Suppose that $\mathbb{F}_2$ is an homomorphic image of S via an homomorphism of rings ϕ. Then, since R is unitary we have that $R \xrightarrow{\phi_{|R}} \mathbb{F}_2$ is onto, against the hypotesis. □

It is interesting to see the problem from another point of view. Suppose that S is a ring of characteristic n and set $U(S) = \{s \in S \mid s \text{ is invertible}\}$. We can consider now the image of the group algebra $\mathbb{Z}_n[U(S)]$ inside S via the morphism that sends every element of $U(S)$ to the corresponding element in S, call it S'. The ring S' is invariant under left multiplication by units of S and it is precisely the set of the elements of S that can be written as the sum of some units. By Theorem 2.37, if S is Von Neumann regular right self-injective, we instantly obtain that S' is Von Neumann regular.

Corollary 2.39. *Let S be a Von Neumann regular right self-injective ring of characteristic n that has no factor isomorphic to $\mathbb{F}_2$ (the same type of result holds if we ask S', and not S, to not have an homomorphic factor isomorphic to $\mathbb{F}_2$). Then every element of S is the sum of two units. In particular, this holds if $n > 0$ and 2 does not divide n.*

Proof. The first part of the result follows immediately from Theorem 2.33 (notice that by Theorem 2.38 the same result holds if we suppose that S' has no homomorphic image isomorphic to $\mathbb{F}_2$). Suppose now that S has charcateristic n and respects the properties in the statement. Now if $\mathbb{F}_2$ is an homomorphic image of S via ϕ and we consider the natural ring morphism $\mathbb{Z} \xrightarrow{i} S$. We have that $f = \phi \circ i$ is a ring homomorphism (remember that for us ring homomorphisms respect the identity), but $2\mathbb{Z} + n\mathbb{Z} \subseteq Ker(f)$ and so $Ker(f) = \mathbb{Z}$, absurd. □

Recall that:

Theorem 2.40. *Every right self-injective ring S is exchange and right quasi-continuous.*

Proof. Every quasi-injective module is exchange and quasi-continuous (Theorems 1.27 and 1.21). The endomorphism ring of an exchange module is an exchange ring, so in particular $R \cong \mathrm{End}(R_R)$ is exchange. □

Theorem 2.40 shows that the next result is a generalization of Theorem 2.33.

Theorem 2.41. *Let M_R be a right quasi-continuous R-module with the finite exchange property and let $S = End(M_R)$. If no factor ring of S is isomorphic to $\mathbb{F}_2$, then every element of S is sum of two units.*

Theorem 2.42. *A continuous module has the exchange property.*

As an observation, we get that the endomorphism ring of every continuous module that has no factor isomorphic to $\mathbb{F}_2$, is such that all its elements are the sum of two units.

Now that we have all this theorems about the unit structure of Von Neumann regular self-injective rings it's time to apply them, keeping in mind Theorem 2.4.

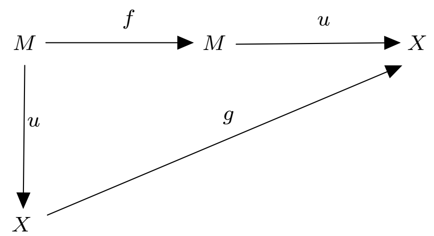
Chapter 3

General properties of automorphism invariant modules

In the last chapter, we proved that for injective and quasi-injective modules the endomorphism ring structure is deeply linked to right self-injective Von Neumann regular rings and to rings in which every idempotent can be lifted modulo the Jacobson radical. In this chapter, we will see how to link all the results about the units structure of a ring, to the problem that was suggested by Theorem 2.4: when can we say that an automorphism-invariant module is also endomorphism-invariant? We will develope a general theory, and to do so we ask the modules we are considering to have certain, really natural, properties: throughout this chapter χ is a class of modules closed under isomorphisms and M a right R-module with a monomorphic χ-envelope $u : M \to X$ such that $\mathrm{End}(X)/J(\mathrm{End(X)})$ is Von Neumann regular right self-injective and idempotents of $\mathrm{End}(X)$ lift modulo the Jacobson radical.

We try at first to find a morphism to include $\mathrm{End}(M)/J(\mathrm{End}(M))$ in $\mathrm{End}(X)/J(\mathrm{End}(X))$ (this will permit us to use to use Corollary 2.38).

By definition of χ-envelope, every endomorphism f of M can be extended to a endomorphism g of X,



and we would like to use this fact to create a monomorphism from $\mathrm{End}(M)/I$ to $\mathrm{End}(X)/J$, where I and J are ideals respectively of $\mathrm{End}(M)$ and $\mathrm{End}(X)$. The problem is that we don't know already if the extensions of f are somehow related.

Lemma 3.1. *Let f be an endomorphism of M, and g_1, g_2 endomorphisms of X such that $g_i \circ u = u \circ f$ for $i = 1, 2$. Then $g_1 - g_2 \in J(End(X))$.*

Proof. To prove the Lemma is enough to show that $1 - h \circ (g_1 - g_2)$ is invertible for any $h \in \mathrm{End}(X)$. But $h \circ (g_1 - g_2) \circ u = h \circ (f - f) = 0$, therefore $(1 - h \circ (g_1 - g_2)) \circ u = u$ and, by definition of χ-envelope, this implies that $1 - h \circ (g_1 - g_2)$ is an automorphism. □

Lemma 3.1 shows that it is possible to define a ring homomorphism $\phi : \mathrm{End}(M) \to \mathrm{End}(X)/J(\mathrm{End}(X))$, therefore it is possible to define an injective ring homomorphism $\bar{\phi} : \mathrm{End(M)}/K \to \mathrm{End}(X)/J(\mathrm{End}(X))$, where $K = Ker(\phi)$. Looking back at Theorem 2.16 it is natural to ask whether K has some particular properties when the module M is χ-automorphism invariant, in particular if it coincides with $J(\mathrm{End}(M))$.

3.1 Structure of automorphism invariant modules

Throughout this section, we will suppose the right R-module M to be χ-automorphism invariant.

In order to solve the problem we anticipated in the previous section we need a Lemma to show that every element h of $J(\mathrm{End}(X))$ can be thought as an extension of some element of K.

Lemma 3.2. *If $h \in J(End)(X)$, there exists $k \in K$ such that $h \circ u = u \circ k$.*

Proof. Since $h \in J(\mathrm{End}(X))$, $1-h$ is invertible. Since M is χ-automorphism invariant, there exists an automorphism f of M such that $(1-h) \circ u = u \circ f$ (Theorem 2.3). Now, since $h = 1 - (1 - h)$, we have that

$$h \circ u = (1 - (1 - h)) \circ u = u - u \circ f = u \circ (1 - f).$$

Since $\phi(1 - f) = h + J(\mathrm{End}(X)) = 0$, $1 - f \in K$. □

Now we are ready to prove that $K = J(\mathrm{End}(M))$, but we want to show something more: we will discover that $\mathrm{End}(M)/J(\mathrm{End}(M))$ is Von Neumann regular and that the idempotents of $\mathrm{End}(M)$ lift modulo $J(\mathrm{End}(M))$.

Theorem 3.3. *The ring End$(M)/K$ is Von Neumann regular, $K = J(End(X))$ and the idempotents of End(M) lift modulo the Jacobson radical.*

Proof. Let $T = \phi(\text{End}(M)/K)$, $S = \text{End}(X)$ and $J = J(\text{End(X)})$. We want to prove first of all that T is invariant under left multiplication by units of S/J in order ro apply Theorem 2.37; notice that once we are able to do so, we automatically obtain that T is Von Neumann regular. Consider an invertible element $g + J \in S/J$. As in the proof of Theorem 2.4, g is an automorphism of X, so, since M is χ-automorphism invariant, we can find an automorphism f of M such that $u \circ f = g \circ u$ ($\bar{\phi}(f + K) = g + J$). Now consider any element $t \in T$. In particular, there exists an element $h + K \in \text{End}(M)/K$ such that $\bar{\phi}(h + K) = t$. Now

$$(g + J) \circ t = (g + J) \circ (\bar{\phi}(h + K)) = \bar{\phi}((f \circ h) + K) \in T$$

therefore T is Von Neumann regular. Since $\text{End}(M)/K$ is Von Neumann regular, $J(\text{End}(M)/K) = 0$ and this implies that $J(\text{End}(M)) \subseteq K$. In order to prove that $K \subseteq J(\text{End}(M))$ it is enough to show that for every element $f \in K$ and every element $r \in \text{End(M)}$, the element $1 - f \circ r$ is right invertible. But since K is a two sided ideal it is enough to show that $1 - f$ is invertible. Let $g \in S$ be such that $g \circ u = f \circ u$. Since $f \in K$, we get that $\bar{\phi}(f + K) = g + J = J$, therefore $1 - g$ is a unit and from Theorem 2.3, we can immediately conclude that $1 - f$ is invertible.

Now it remains to prove the last part of the theorem, the property of the lifting of idempotents modulo the Jacobson radical.

Consider an element $f + K \in \text{End}(M)/K$ such that $f + K = f^2 + K$ and extend it to an endomorphism g of S such that $u \circ f = g \circ u$. As $g + J$ is the image of an idempotent element, it is clear that $g + J$ is idempotent, and since the idempotents of S lift modulo J, there exists an idempotent $e \in S$ such that $g - e \in J$. From Lemma 3.2, we get that there exists an element $k \in K$ such that $(g - e) \circ u = u \circ k$, therefore $u \circ (f - k) = e \circ u$ and $\phi(f - k) = e$. This implies that $u \circ (f - k)^2 = e \circ u \circ (f - k) = e^2 \circ u = e \circ u = u \circ (f - k)$, and since u is monic we get that $(f - k)$ is an idempotent.

□

The next Corollary shows that M satisfies the finite exchange property.

Corollary 3.4. *The module M has the finite exchange property.*

Proof. It is a simple consequence of Theorem 3.3, Theorem 1.31, the fact that every Von Neumann regular ring has the exchange property and Theorem 1.28. □

Theorem 3.5. *If End(M) has no factor isomorphic to* $\mathbb{F}_2$*, then M is* χ*-endomorphism invariant and therefore*

$$End(M)/J(End(M)) \cong End(X)/J(End(X))$$

In particular, $End(M)/J(End(M))$ *is Von Neumann regular right self-injective and idempotents lift modulo the Jacobson radical of* $End(M)$*.*

This is the case when $char(End(M)) = n > 0$ *and* n *is not a multiple of 2.*

Proof. During the proof of Theorem 3.3 we showed that

$$T = \mathrm{End}(M)/J(\mathrm{End}(M))$$

is stable under left multiplication by units of $S = \mathrm{End}(X)/J(\mathrm{End}(X))$. We can now apply Theorem 2.38 to obtain the first part of the theorem (the fact that M is χ-endomorphism invariant is then guaranteed by Theorem 2.4). For the second part it suffices to apply Corollary 2.39. □

Corollary 3.6. *Let R be a commutative ring with no factor ring isomorphic to* $\mathbb{F}_2$*. Then M is* χ*-endomorphism invariant.*

Proof. It suffices to show that $\mathrm{End}(M)$ does not have an homomorphic image isomorphic to $\mathbb{F}_2$. But that is easy: since R is commutative we have a ring homomorphism $R \to \mathrm{End}(M)$, and if $\mathrm{End}(M)$ has an homomorphic image isomorphic to $\mathbb{F}_2$, then also R has one. □

In particular, Theorem 3.5 shows that an automorphism-invariant module N, with endomorphism ring $\mathrm{End}(N)$ such that $\mathrm{End}(N)$ has no homomorphic image isomorphic to $\mathbb{F}_2$, is quasi-injective.

Corollary 3.6 implies instead that if the base ring R we are considering is commutative and has no factor isomorphic to $\mathbb{F}_2$, then any automorphism-invariant module is quasi-injective. Hence, over every field with prime characteristic different from two, any automorphism-invariant module is quasi-injective.

The following theorem is a key result that will be used in the next chapters. In particular, it says that an automorphism-invariant indecomposable module N that is not quasi-injective, is such that $\mathrm{End}(N)/J(\mathrm{End}(N)) \cong \mathbb{F}_2$.

Theorem 3.7. *Let M be indecomposable and suppose that M is not* χ*-endomorphism invariant.*

Then $End(M)/J(End(M)) \cong \mathbb{F}_2$ *and* $End(X)/J(End(X))$ *has an homomorphic image isomorphic to* $\mathbb{F}_2 \times \mathbb{F}_2$*.*

3.2 Exchange property for automorphism invariant modules

As in the previous section, the module M is supposed to be χ-automorphism invariant. We have already proved that a module of this kind satisfies the finite exchange property. In this section, we study some particular cases in which the full exchange property holds.

The following theorem shows that $\text{End}(M)/J(\text{End}(M))$ (in some special cases) can be factorized as the product of a Boolean ring and a Von Neumann regular right self-injective one. If we suppose that $\text{End}(M)/J(\text{End}(\text{M}))$ has no factor isomorphic to $\mathbb{F}_2$, then the Boolean part disappears. In fact it is possible to show that every Boolean ring has characteristic two, and that every Boolean domain is isomorphic to $\mathbb{F}_2$; if we consider then a maximal ideal I of a Boolean ring S we have that $S/I \cong \mathbb{F}_2$ (this gives another way to look at Theorem 3.5).

Before continuing we have to fix some notations and some definitions. If N and L are two direct summands of M we will say that $\text{Hom}(N, L) \subseteq \text{End}(M)$ meaning that the application which sends $f \in \text{Hom}(N, L)$ to

$$\iota_L \circ f \circ \pi_N \in \text{End(M)}$$

is injective. We say that a ring R is *semiboolean*, if $R/J(R)$ is a Boolean ring, and we say that a module is *semiboolean*, if its endomorphism ring is semiboolean. We say that a right R-module N is *square-free* if for every right R-module $X \neq 0$ there is not a monomorphism $X \oplus X \to N$ (we can say that N does not contain squares).

Theorem 3.8. *Suppose that M is such that each of its direct summands has a χ-envelope. Then M admits a decomposition $M = N \oplus L$ such that:*

(a) The module N is semiboolean.

(b) The module L is χ-endomorphism invariant (it can be proved that every element of End(L) is the sum of two units), End(L)/J(End(L)) is Von Neumann regular right self-injective and idempotents lift modulo J(End(L)).

(c) Both $\text{Hom}_R(N, L)$ and $\text{Hom}_R(L, N)$ are contained in End(M).

In particular, End(M)/J(End(M)) is the direct product of a right self-injective Von Neumann regular ring and a Boolean one.

Theorem 3.9. *Suppose that N is a square-free module with the finite exchange property. Then N has the exchange property.*

The next theorem is the key result of this section. In particular, it says that automorphism-invariant modules satisfy the full exchange property.

Theorem 3.10. *Assume that every direct summand of M has a χ-envelope, and suppose that, for χ-endomorphism invariant modules, the finite exchange property implies the full exchange property. Then M has the full exchange property.*

Proof. Let's use Theorem 3.8 to obtain a direct sum decomposition $M = N \oplus L$, where L is χ-endomorphism invariant and N is semiboolean. By Corollary 3.4, and the hypotesis it follows immediately that L has the exchange property. Since N is semiboolean and $\mathrm{End}(N)$ has the property of the lifting of idempotents modulo its Jacobson radical, it follows that N is square-free. Being a direct summand of M, N is χ-automorphism invariant, hence it has the finite exchange property, so, by Theorem 3.9, we can conclude. □

3.3 Properties of automorphism invariant modules

In this section, we suppose M to be χ-automorphism invariant.

We want to analyze some more properties of automorphism-invariant modules. Firstly we deal with an analogue of Theorem 1.18. In particular, we will see that asking X or M to be indecomposable some nice properties arise.

Theorem 3.11. *If X is indecomposable then M is χ-endomorphism invariant.*

Proof. If X is indecomposable it means that $\mathrm{End}(X)/J(\mathrm{End}(X))$ has no non-trivial idempotents (idempotents in $\mathrm{End}(X)$ lift). But $\mathrm{End}(X)/J(\mathrm{End}(X))$ is a Von Neumann regular ring with no non-trivial idempotents, therefore it is a division ring, and this implies that every element of $\mathrm{End}(X)$ is a sum of units. □

Notice that if N is an automorphism-invariant module with indecomposable injective envelope $E(N)$, then N is quasi-injective. We can also say something more: suppose that N is a quasi-injective indecomposable module, then from Theorem 1.18, we get that $E(N)$ is indecomposable; so in particular if N is automorphism-invariant and indecomposable, then $E(N)$ is indecomposable if and only if N is quasi-injective. It is then natural the following:

Theorem 3.12. *Suppose now that M is indecomposable. Then the following are equivalent:*

(a) The module M is χ-endomorphism invariant.

(b) The module X is indecomposable.

As we already noticed: if N is automorphism-invariant and indecomposable, then $E(N)$ is indecomposable if and only if N is quasi-injective .

We will say that a ring R is *clean*, if every element $a \in R$ can be written as $a = e + u$ where $e \in R$ is an idempotent and $u \in R$ is a unit. We say that a module N is *clean*, if its endomorphism ring is clean.

We will see, in Theorem 3.15, that every automorphism-invariant module is clean.

Theorem 3.13. *Any right self-injective ring S is clean.*

Theorem 3.14. *Let S be a ring such that $S/J(S)$ is clean and suppose that S has the property of the lifting of idempotents modulo its Jacobson radical. Then S is clean.*

Theorem 3.15. *Assume that every direct summand of M has a χ-envelope. Then M is a clean module.*

Proof. From Theorem 3.8, we get that $S = \mathrm{End}(M)/J(\mathrm{End}(M))$ can be written as the product of a Boolean ring and a right self-injective one. So S is the product of two clean rings, therefore it is clean. Since idempotents lift modulo the jacobson radical we have, applying Theorem 3.14, that $\mathrm{End}(M)$ is clean. □

Consider now an indecomposable automorphism-invariant module N. We know that N satisfies the exchange property, and so, by Theorem 1.32, the endomorphism ring of N is local. Notice that this result can be obtained also as a Corollary of Theorem 3.15.

Corollary 3.16. *Let N be an indecomposable automorphism-invariant module. Then every element of End(N) is a unit, or it is the sum of the identity morphism and a unit. Hence End(N) is a local ring with the property of the lifting of idempotents.*

We say that a ring S is *unit regular*, if for every $a \in S$ there exists a $u \in U(R)$ such that $a = aua$. We now want to focus our attention on directly finite modules to show that when M is directly finite, then $\mathrm{End}(M)/J(\mathrm{End}(M))$ is unit regular.

Theorem 3.17. *Every directly finite right self-injective Von Neumann regular ring is unit regular.*

We need now a Lemma to prove the result we previously announced. In particular, we need the fact that M and X are directly finite whenever one of them is.

Lemma 3.18. *The module M is directly finite if and only if X is directly finite.*
The result holds in particular if M or X have finite Goldie dimension.

We are now ready to prove our result:

Theorem 3.19. *Assume that every direct summand of M has a χ-envelope and suppose that M is directly finite. Then $End(M)/J(End(M))$ is unit regular.*

Proof. We can apply Theorem 3.8 to obtain a decomposition of the ring $\text{End}(M)/J(\text{End}(M))$ into the product of a Boolean ring S_1 and a Von Neumann regular right self-injective one S_2 associated to a decomposition

$$M = N \oplus L.$$

From Theorem 3.17, we get that S_2 is unit regular (clearly S_2 is directly finite otherwise we would obtain that L is not directly finite and so M). Since every Boolean ring is unit regular, and since the product of two unit regular rings is unit regular, we obtain that $\text{End}(M)/J(\text{End}(M))$ is unit regular. □

In particular, if N is an automorphism-invariant directly finite module, then $\text{End}(N)/J(\text{End}(N))$ is unit regular.

We now want to introduce some more definitions.

We say that a right R-module N has the *cancellation property* if every time we have $N \oplus A \cong N \oplus B$, for some right R-modules A and B, then $A \cong B$. We say instead that N has the *internal cancellation property* if everytime $N = A_1 \oplus A_2 \cong B_1 \oplus B_2$, for some submodules A_1, A_2, B_1, B_2 of M, and $A_1 \cong B_1$, we have that $A_2 \cong B_2$.

We say that a module A has the *substitution property* if for every module N such that $N = A_1 \oplus H = A_2 \oplus K$, where $A_1 \cong A_2 \cong A$,there exists a submodule A' of N such that $N = A' \oplus H = A' \oplus K$ (notice that in this case $A' \cong A$).

With those definitions in mind it is possible to prove the following:

Theorem 3.20. *Suppose that every direct summand of M has a χ-envelope. Then the following are equivalent:*

(a) The module M is directly finite.

(b) The module M has the internal cancellation property.

(c) The module M has the cancellation property.

(d) The module M has the substitution property.

(e) The module X is directly finite.

(f) The module X has the internal cancellation property.

(g) The module X has the cancellation property.

(h) The module X has the substitution property.

Notice that an automorphism-invariant module with finite Goldie dimension has all the properties stated in Theorem 3.20.

Chapter 4

Automorphism-invariant modules

In this chapter, we study some properties of automorphism-invariant modules.

In particular, we will see that pseudo-injective modules coincide with the automorphism-invariant ones. We will then apply the results obtained in the previous chapters in the framework of automorphism-invariant modules.

For completeness we recall here some definitions we have used throughout this thesis. When we consider the class χ of injective modules we say that a module is endomorphism-invariant when it is χ-endomorphism invariant, and we say that a module is automorphism-invariant if it is χ-automorphism invariant.

Theorem 4.1. *Let M be a right R-module and let $(i, E(M))$ be its injective envelope. Then the following are equivalent:*

(a) The module M is automorphism-invariant.

(b) Every isomorphism between two essential submodules of M can be extendend to an endomorphism of M.

Proof. (a)⇒(b). Suppose that N, N' are two essential submodules of M and suppose that there exits an isomorphism $N \xrightarrow{f} N'$. By the usual properties of injective modules, we can extend f to an endomorphism f' of $E(M)$. Since f' coincides with f on N and N is essential in M (and so in $E(M)$) we get that $Ker(f') \cap M = 0$, therefore f' is a monomorphism. So $f'(E(M)) \cong E(M)$ is an injective module contained in $E(M)$, so it is a direct summand of $E(M)$. As $f(N) = N' \subseteq f'(E(M))$ is essential in M (and so in $E(M)$), we can conclude.

(b)$\Rightarrow$(a) Let f be an automorphism of $E(M)$, let $N = M \cap f(M)$ and let $N' = f^{-1}(N)$. Since f is an automorphism we have that $f(M) \leq_e E(M)$, therefore $N \leq_e M$. By the previous consideration, we get immediately that N' is essential in M, so, by the hypotesis, we can extend the isomorphism between N and N' to an endomorphism g of $E(M)$. We want now to show that $(f-g)(M) = 0$ by contraddiction. Suppose then that $(f-g)(M) \neq 0$. Since $M \leq_e E(M)$ we have that $(f-g)(M) \cap M \neq 0$, therefore there exist $m, m' \in M$ such that $(f-g)(m) = m'$. This implies that $f(m) = g(m) + m'$, so $m \in N'$ and $m' = 0$, contraddiction. □

We say that a module M is *pseudo-injective* if every monomorphism from a submodule of M into M can be extended to an endomorphism of M. More formally: we say that a right R-module M is pseudo-injective if, for every submodule $N \leq M$ and every monomorphism $f : N \to M$, there exists an endomorphism g of M such that $f = g_{|N}$.

It is clear, prooceding in the usual manner, that pseudo-injective modules are automorphism-invariant, but proving the converse is more technical (a proof can be done using Theorem 4.1), so we will not include the proof which can be found in [5].

Theorem 4.2. *A module M is automorphism-invariant if and only if it is pseudo-injective.*

For completeness we recall here a result that was proved in a more general case.

Theorem 4.3. *Let M be a right R-module such that End(M) has no factor isomorphic to $\mathbb{F}_2$. Then M is quasi-injective if and only if it is automorphism-invariant.*

Proof. It's a simple consequence of Theorem 3.5. □

Here we collect two results which can be really useful.

Theorem 4.4. *Let M be a right R-module such that its endomorphism ring has no factor isomorphic to $\mathbb{F}_2$, and let $E(M)$ be its injective envelope. Then End$(E(M))$ has no factor isomorphic to $\mathbb{F}_2$.*

Proof. Suppose that $S = \text{End}(E(M))$ has a factor isomorphic to $\mathbb{F}_2$, while End(M) has not. Then, there exists a ring homomorphism $S \xrightarrow{\phi} \mathbb{F}_2$, but since $\mathbb{F}_2 \cong \text{End}(\mathbb{F}_2)$, $\mathbb{F}_2$ inherits an S-module structure. As $\mathbb{F}_2$ is simple as a $\mathbb{Z}$-module, it is surely simple as an S-module (every S-submodule is also a $\mathbb{Z}$-submodule). Then ϕ is an S-module endomorphism, and being $\mathbb{F}_2$

simple, we obtain that $J(S)$ must be contained in the kernel of ϕ. Hence ϕ factorizes through a morphism $\phi' : S/J(S) \to \mathbb{F}_2$. Since $E(M)$ is injective, we obtain that the every endomorphism $f \in \text{End}(M)$ can be extended to an endomorphism of $E(M)$, let's call it ϕ_f. From Lemma 3.1, we obtain a ring homomorphism $\text{End}(M) \xrightarrow{\psi} S/J(S)$ that sends every endomorphism f of M to $\psi_f + J(S)$. Now, $\phi' \circ \psi$ is a ring homomorphism $\text{End}(M) \to \mathbb{F}_2$, a contraddiction. □

Corollary 4.5. *Let M be a right R-module such that End(M) has no factor isomorphic to $\mathbb{F}_2$. Then End$(E(M))$ is a clean ring in which every element is the sum of some units.*

Proof. From Theorem 4.4, we get that $\text{End}(E(M))/J(\text{End}(E(M)))$ has no factor isomorphic to $\mathbb{F}_2$. Since $E(M)$ is injective, $\text{End}(E(M))/J(\text{End}(E(M)))$ is Von Neumann regular right self-injective. Hence, from Theorem 2.33 and the usual techniques, we can conclude. □

Theorem 4.6. *Let M be a continuous right R-module and let $S =$ End(M). Then, every element of S is the sum of two units if and only if S has no factor ring isomorphic to $\mathbb{F}_2$.*

Proof. It's a consequence of Theorem 2.42 and Theorem 2.41. □

We are now ready to give some important results about algebras over fields.

We already proved (Theorem 3.6) that if R is a commutative ring with no factor isomorphic to $\mathbb{F}_2$, the following are equivalent for a right R-module:

(a) The module M is automorphism-invariant.

(b) The module M is quasi-injective.

We can ask what happens if we have an A-module, with A an $\mathbb{F}$-algebra. In order to study the problem we need to introduce the following:

Lemma 4.7. *Let R be a ring, and let S be a unitary subring of the center of R. If $\mathbb{F}_2$ does not admit a right S-module structure, then, the endomorphism ring of each right R-module M has no factor isomorphic to $\mathbb{F}_2$.*

Proof. Since S is commutative, the application $\phi : S \to \text{End}(M_R)$ which sends an element $s \in S$ to the morphism $\phi_s \in \text{End}(M_R)$ defined by $\phi_s(m) = ms$, is a ring homomorphism. If $\text{End}(M_R)$ has some factor isomorphic to $\mathbb{F}_2$ through a ring homomorphism ψ, then composing ϕ and ψ we get that S has a factor isomorphic to $\mathbb{F}_2$ (so $\mathbb{F}_2$ would have an S-module structure), absurd. □

Theorem 4.8. *Suppose that A is an algebra over a field $\mathbb{F}$ with more than two elements. Then a right A-module M is automorphism-invariant if and only if it is quasi-injective.*

Proof. From Lemma 4.7, we get that the endomorphism ring of every right A-module M has no factor isomorphic to $\mathbb{F}_2$ ($\mathbb{F}_2$ does not admit a $\mathbb{F}$-module structure), and by Theorem 4.3 we can conclude. □

Corollary 4.9. *Let A be an algebra over a field $\mathbb{F}$ with more than two elements and suppose that A is right automorphism-invariant, then A is right self-injective.*

From Theorem 4.8, we obtain in particular that the polynomial rings over a field with more than two elements are such that every automorphism-invariant module over them is quasi-injective.

4.1 Automorphism invariant modules with finite Goldie dimension

In this section, we present some results about pseudo-injective modules and we apply them in the study of modules with finite Goldie dimension.

Lemma 4.10. *If M is automorphism-invariant, every direct summand of M is automorphism-invariant.*

Proof. Theorem 2.5. □

Lemma 4.11. *If $M = M_1 \oplus M_2$ is automorphism-invariant then M_1 is M_2-injective and M_1 is M_2-injective.*

Proof. We want to prove that M_1 is M_2-injective. Let A be a submodule of M_2 and let $f : A \to M_1$ be a morphism of right R-modules. We know that A has a $\cap$-complement B in M_2. Let's call $C = A \oplus B$. We can extend f to an homomorphism $g : C \to M_1$ letting $g(B) = 0$. Now, let $\alpha : M_1 \oplus C \to M_1 \oplus M_2$ be the module monomorphism which sends an element $(x, c) \in M_1 \oplus C$ to the element $(x + g(c), c)$. Since $\alpha(M_1 \oplus M_2) = M_1 \oplus C$ we can use Theorem 4.1 to extend α to an endomorphism β of $M_1 \oplus M_2$. Call ι_2 the natural inclusion of M_2 inside $M_1 \oplus M_2$ and call π_1 the projection of $M_1 \oplus M_2$ onto M_1. The morphism we were looking for is then $\pi_1 \circ \beta \circ \iota_2$. □

Theorem 4.12. *Let M be an automomorphism-invariant module such that $M = M_1 \oplus M_2$. Then M_1, M_2 are automorphism-invariant, M_1 is M_2-injective and M_1 is M_2-injective.*

Proof. It's a simple consequence of Lemmas 4.10, 4.11. □

One can prove that:

Theorem 4.13. *A finite direct sum of automorphism-invariant modules is automorphism-invariant if and only if the direct summands are relatively injective (this simply means that every direct summand is injective relatively to the others).*

Suppose that we have an automorphism-invariant module M with finite Goldie dimension. Then M can be written as a direct sum of relatively injective automorphism-invariant indecomposable modules with finite Goldie dimension, let's say $M = \oplus_{i=1}^{n} M_i$. If M is quasi-injective then all the M_i are quasi-injective, being direct summands of M. This means, as the M_i are indecomposable, that the M_i's are uniform (Theorem 3.12 and Theorem 1.6). If the M_i's are quasi-injective (notice that the M_i's are relatively injective), then M is quasi-injective (Theorem 1.10 and Theorem 1.12) and the M_i's are uniform.

These considerations imply that the study of the finite Goldie dimension automorphism-invariant modules that are quasi-injective reduces to the study of uniform quasi-injective modules.

The following example shows that not every autmorphism-invariant module with finite Goldie dimension is endomorphism-invariant:

Example 4.1.1. Let $R = \begin{bmatrix} \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & \mathbb{F}_2 & 0 \\ 0 & 0 & \mathbb{F}_2 \end{bmatrix}$ and let $M = \begin{bmatrix} \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. As $M = e_{11}R$, where $e_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a primitive idempotent (i.e. cannot be written as the sum of two ortoghonal idempotents), the module M is indecomposable. Since R is a finite dimensional $\mathbb{F}_2$-algebra, M is an artinian module and hence it has finite Goldie dimension. It can be checked that the only automorphism of $E(M)$ is the identity, hence M is automorphism-invariant. But M is not uniform: in fact it has two simple submodules $e_{1,2}R$ and $e_{1,3}R$, where $e_{1,2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $e_{1,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore M cannot be quasi-injective.

Theorem 4.14. *Let M be an indecomposable automorphism-invariant module with finite Goldie dimension and suppose that M is not quasi-injective. Then:*

(a) *End*$(M)/J($*End*$(M)) \cong \mathbb{F}_2$

(b) *There exist a finite set of non isomorphic indecomposable injective modules* $\{E_i\}_{i=1,\dots,n}$*, with* $n \geq 2$*, such that*

$$E(M) = \oplus_{i=1}^{n} E_n$$

and *End*$(E_i)/J($*End*$(E_i)) \cong \mathbb{F}_2$ *for every* $i = 1, \dots, n$*.*

Proof. The first part of the proof is a consequence of Theorem 3.7. It remains to prove the second part.

From Theorem 3.8, we get that M can be written as the direct sum of a semiboolean module and an endomorphism-invariant one. Since M is indecomposable and not quasi-injective it follows that M is semiboolean. As M has finite Goldie dimension, $E(M) = \oplus_{i=1}^{n} E_i$, where E_i s an injective indecomposable module for every $i = 1, \dots, n$, and $n \geq 2$ (otherwise M would be quasi-injective by Theorem 3.12). It can be easily shown that $\mathrm{End}(E)/J(\mathrm{End}(E))$ is boolean. Since $\mathrm{End}(E)/J(\mathrm{End}(E))$ is Boolean, E is square-free, hence, the E_i's are two by two non isomorphic and

$$\mathrm{End}(E_i)/J(\mathrm{End}(E_i))$$

is Boolean for every $i = 1, \dots, n$. As the E_i's are indecomposable it follows that $\mathrm{End}(E_i)/J(\mathrm{End}(E_i))$ is a Boolean, division ring for every $i = 1, \dots, n$, hence $\mathrm{End}(E_i)/J(\mathrm{End}(E_i)) \cong \mathbb{F}_2$ for every $i = 1, \dots, n$. □

4.2 Noetherian rings

The goal of this section is to prove that over a commutative noetherian ring R, any automorphism-invariant module is quasi-injective.

We say that a right R-module M is *finitely cogenerated* if $\mathrm{soc}(M)$ is finitely generated and essential in M.

From Theorem 4.14, we get immediately the following:

Theorem 4.15. *Let* M *be an indecomposable finitely cogenerated automorphism-invariant module which is not quasi-injective. Since the socle of* M *is finitely generated we can write* *soc*$(M) = \oplus_{i=1}^{n} S_i$*, where the* S_i *are simple modules. Then:*

(a) $n \geq 2$*.*

(b) *The ring* *End*$(M)/J($*End*$(M))$ *is isomorphic to* $\mathbb{F}_2$*.*

(c) The ring End(S_i) is isomorphic to $\mathbb{F}_2$ for every $i = 1, \dots, n$.

(d) The S_i's are two by two non isomorphic.

Now, in order to prove the main result of this section, we need a technical Lemma that extends Theorem 4.15. Before giving the Lemma we need some definitions.

We say that a ring R is *right bounded* if each essential right ideal of R contains a two sided ideal with the property of being essential as a right submodule of R. It can be proved that a right noetherian ring R is right bounded if and only if every essential right ideal of R contains a non-zero two sided ideal.

Lemma 4.16. *Let R be a right bounded right noetherian ring and M a semiboolean automorphism-invariant module that is not quasi-injective. Then, there exist simple submodules $S_i \leq M$, with $i \in I$ and $2 \leq |I|$, such that soc(M) $= \oplus_{i \in I} S_i$ is essential in M, End(S_i) $\cong \mathbb{F}_2$ for every $i \in I$ and the S_i's are two by two non isomorphic. Moreover, if R is commutative, then $|S_i| = 2$ for every $i \in I$.*

Notice that if we ask R to be right bounded and right noetherian, and we consider an automorphism-invariant indecomposable module M that is not quasi-injective, then M is semiboolean and we can apply Lemma 4.16 to it (it is a consequence of Theorem 3.8).

Theorem 4.17. *Let R be a commutative noetherian ring with no homomorphic images isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$. Then any automorphism-invariant module is quasi-injective.*

Proof. We want to act by contraddiction, hence we suppose that M is not quasi-injective. From Theorem 3.8, we can write $M = N \oplus L$ where N is semiboolean, any element of $\mathrm{End}(L)/J(\mathrm{End}(L))$ can be written as the sum of two units and $\mathrm{End}(M)/J(\mathrm{End}(M)) \cong \mathrm{End}(N)/J(\mathrm{End}(N)) \times \mathrm{End}(L)/J(\mathrm{End}(L))$. Since N is a direct summand of M, then, by Theorem 5.4, we get that N is automorphism-invariant. Since L is quasi-injective, we get that N cannot be quasi-injective otherwise M would be quasi-injective, against the hypotesis. From Lemma 4.16, we deduce that there exist simple submodules $S_i \leq N$, with $i \in I$ and $2 \leq |I|$, such that $\mathrm{soc}(N) = \oplus_{i \in I} S_i$ is essential in N, $\mathrm{End}(S_i) \cong \mathbb{F}_2$ for every $i \in I$ and the S_i's are two by two non isomorphic. Since we are in the commutative case we also get that $|S_i| = 2$ for every $i \in I$. Now, write $S_i = \{0, s_i\}$, and define $p_i : R \to S_i$ as the morphism $p_1(1_R) = s_i$. Since the p_i's are surjective and the module S_i are simple we get that $N_i = Ker(p_i)$ is a maximal ideal of R for every $i \in I$. Since $|I| \geq 2$

we can find at least two indexes $i, j \in I$ and we can define the ring homomorphism $p : R \to R/N_i \times R/N_j$ by $p(1) = p_i(1) \times p_j(1)$. Since R/N_i and R/N_j have just two elements we get that $R/N_i \times R/N_j \cong \mathbb{F}_2 \times \mathbb{F}_2$. As S_i and S_j are non-isomorphic, N_i and N_j are two different maximal ideals of R, hence by Chinese Remainder Theorem we get that p is surjective against the hypotesis. □

The idea to prove that every automorphism-invariant module over a commutative noetherian ring is quasi-injective, is to show that being automorphism-invariant is a local property.

Corollary 4.18. *Let R be a commutative noetherian ring. Then any automorphism-invariant right R-module is endomorphism-invariant.*

Proof. The ring R cannot have an homomorphic image isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$, otherwise it would have two maximal ideals, against locality. From Theorem 4.17, we can conclude. □

Theorem 4.19. *Let R be a commutative noetherian ring and M an R-module. Then the following are equivalent:*

(a) The module M is automorphism-invariant.

(b) The module $M_{\mathfrak{p}}$ is an automorphism-invariant as an $R_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$ of R.

(c) The module $M_{\mathfrak{m}}$ is automorphism-invariant as an $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$.

Also being a quasi-injective module over a commutative noetherian ring is a local property:

Theorem 4.20. *Let R be a commutative noetherian ring and M an R-module. Then the following are equivalent:*

(a) The module M is quasi-injective.

(b) The module $M_{\mathfrak{p}}$ is an quasi-injective as a $R_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$ of R.

(c) The module $M_{\mathfrak{m}}$ is quasi-injective as a $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$.

We are now ready to prove the result we announced at the beginning of this section:

Theorem 4.21. *Let R be a commutative noetherian ring.*
Then any automorphism-invariant module is quasi-injective.

Proof. Since M is automorphism-invariant, $M_{\mathfrak{p}}$ is automorphism-invariant as an $R_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$ of R (Theorem 4.19). Since $R_{\mathfrak{p}}$ is local we obtain, applying Corollary 4.18 that $M_{\mathfrak{p}}$ is quasi-injective as an $R_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$ of R. Finally, from Theorem 4.20, we obtain that M is quasi-injective. □

Chapter 5

Automorhism-coinvariant modules

In this chapter, we discuss the "dual" notion of injective envelopes, *projective covers.*

5.1 Projective covers

We want to look for the smallest possible rapresentation of a module as an homomorphic image of a projective module. Let R be a ring and M be a right R-module. A projective cover is a pair (P, p) where P is a projective right R-module and $p : P \to M$ is a superfluous epimorphism. Analogously to the injective case we get the following:

Theorem 5.1 (Fundamental Lemma of projective covers)**.** *Let (P, p) be a projective cover of a right R-module M. If Q is a projective module and $q : Q \to M$ is an epimorphism, then Q has a direct sum decomposition $Q = P' \oplus P''$ where $P' \cong P$, $P'' \subseteq Ker(q)$ and $(P', q|_{P'})$ is a projective cover of M.*

Theorem 5.2 (Uniqueness of projective covers up to isomorphism)**.** *Projective covers, when they exist, are unique up to unique isomorphism in the following sense. If (Q, q) and (P, p) are two projective covers of M, then there exists an isomorphism h such that $p \circ h = q$.*

We say that a module M, with projective cover (P, p), is *endomorphism-coinvariant* if any endomorphism f of P factorizes through p, namely, if there exists an endomorphism g of M such that $p \circ f = g \circ p$.

We say that a module M, with projective cover (P, p), is *automorphism-coinvariant* if any automorphism f of P factorizes through p, namely, if there exists an endomorphism g of M such that $p \circ f = g \circ p$.

One can show that there is an equivalent of Theorem 1.16 in the projective case:

Theorem 5.3. *A module M that admits a projective cover is endomorphism-coinvariant if and only if M is quasi-projective.*

There is one fundamental difference between injective envelopes and projective covers: while the first exist for every module, the second does not always exist. Hence, it is not possible to expect that we will have an analogue of all the results we proved in the injective case without asking some particular properties for the ring R involved.

Example 5.1.1. As we have already noticed in Example 1.7.1, $\mathbb{Z}/4\mathbb{Z}$ is a quasi-projective $\mathbb{Z}$-module that is not projective. It is straightforward to notice that $\mathbb{Z}/4\mathbb{Z}$ does not have a projective-cover (since $\mathbb{Z}$ is indecomposable the only possible projective cover of $\mathbb{Z}/4\mathbb{Z}$ would be $(\mathbb{Z}, \pi_4)$, where π_4 is the natural projection of $\mathbb{Z}$ over $\mathbb{Z}/4\mathbb{Z}$, but π_4 is not a small epimorphism).

It is now the time to do what we did in the injective case: generalize the notion of " cover".

5.2 Generalizations of the concept of cover

Let M be a right R-module and let χ be a class of modules closed under isomorphisms. We say that an homomorphism $p : X \to M$, where X is an element of χ, is a *χ-precover* of M, if any other homomorphism $g : X' \to M$, where X' is an element of χ, factorizes through p, namely, if there exists an homomorphism $q : X' \to X$ such that $p \circ q = g$. We say that an χ-precover of M, $p : X \to M$, is a *χ-cover* if whenever there is an endomorphism $g : X \to X$ such that $p \circ g = p$, then g is an automorphism.

Hence, it is natural to give a notion of invariance under endomorphisms and automorphisms.

We say that a module M with a χ-cover $p : X \to M$ is *χ-endomorphism coinvariant* if for any endomorphism g of X there exists an endomorphism f of M such that $f \circ p = p \circ g$.

When χ is the class of projective modules, then the χ-endomorphism invariant modules are quasi-projective.

Anologously, we say that a module M with a χ-cover $p : X \to M$ is *χ-automorphism coinvariant* if for any automorphism g of X there exists an endomorphism f of M such that $f \circ p = p \circ g$.

It can be proved that:

Theorem 5.4. *If M is χ-automorphism coinvariant and every direct summand of M has a χ-cover, then any direct summand of M is χ-automorphism coinvariant.*

We say that a χ-cover $p : X \to M$ of M is epimorphic if p is an epimorphism. Similarly to the injective case, if we consider a module M with an epimorphic χ-cover, then M is χ-automorphism coinvariant precisely when p induces a group isomorphism $\Delta' : \mathrm{Aut}(M) \cong \mathrm{Aut}(X)/\mathrm{coGal}(X)$, where $\mathrm{coGal}(X)$ is the set of the automorphisms g of X such that $p \circ g = p$. We call $\mathrm{coGal}(X)$ the coGalois group of the cover p.

5.3 Properties of covers

In this section, we list some theorems which are the perfect equivalent of the theorems given for χ-envelopes.

In this section, we suppose that χ is a class of modules closed under isomorphisms, M a module with an epimorphic χ-cover $p : X \to M$ such that $\mathrm{End}(X)/J(\mathrm{End}(X))$ is Von Neumann regular, right self-injective and the idempotents of $\mathrm{End}(X)$ lift modulo the Jacobson radical.

Similarly to the injective case, for every endomorphism f of M, we can find an endomorphism g of X such that $p \circ g = f \circ p$. It can be shown that the application $\phi : End(M) \to S/J(S)$ defined by $\phi(f) = g + J(S)$, for every $f \in \mathrm{End}(M)$, is well-defined, and it is an homomorphism of rings.

Now, we give an equivalent of Theorem 3.3:

Theorem 5.5. *If M is χ-automorphism coinvariant, then $K = J(End(M))$, $End(M)/J(End(M))$ is Von Neumann regular and the idempotents of $End(M)$ lift modulo the Jacobson radical.*

It is natural to ask whether in the projective case we can say something about the finite exchange property.

Corollary 5.6. *Let M be χ-automorphism coinvariant. Then M satisfies the finite exchange property.*

The next question that arises is wheter in some special cases a χ-automorphism coinvariant module is also χ-endomorphism coinvariant.

Theorem 5.7. *Let M be a χ-automorphism coinvariant module and assume that End(M) has no homomorphic images isomorphic to $\mathbb{F}_2$. Then M is χ-endomorphism coinvariant and End$(M)/J($End$(M)) \cong$ End$(X)/J($End$(X))$. In particular, End$(M)/J($End$(M))$ is Von Neumann regular right self-injective and the idempotents of End(M) lift modulo the Jacobson radical. This is the case when $n = char($End$(M)) > 2$ and 2 does not divide n.*

As we did in the injective case we are now interested in whether there exists some nice decomposition of $\text{End}(M)/J(\text{End}(M))$ when M is χ-automorphism coinvariant and every direct summand of M has a χ-envelope.

Theorem 5.8. *Let M be χ-automorphism coinvariant and suppose that every direct summand of M has a χ-cover. Then M admits a decomposition $M = N \oplus L$ such that:*

(a) The module N is semiboolean.

(b) The module L is χ-endomorphism coinvariant, End$(L)/J($End$(L))$ is Von Neumann regular right self-injective and the idempotents of End(L) lift modulo the Jacobson radical.

(c) Both Hom$_R(N, L)$ and Hom$_R(L, N)$ are contained in $J($End$(M))$. In particular, End$(M)/J($End$(M)) \cong$ End$(N)/J($End$(N)) \times$End$(L)/J($End$(L))$ is the product of a semiboolean ring and a Von Neumann regular right self-injective one.

What about the exchange property?

Theorem 5.9. *Let M be χ-automorphism coinvariant and suppose that every direct summand of M has a χ-cover. Suppose furthermore that for χ-endomorphism coinvariant modules, the finite exchange property implies the full exchange property. Then M satisfies the full exchange property.*

The natural question that arises now is: is there an analogous of Theorem 3.15? The answer is affirmative by the following theorem:

Theorem 5.10. *Let M be a χ-automorphism coinvariant module and suppose that every direct summand of M has a χ-cover. Then M is a clean module.*

One can also show that for a directly finite χ-automorphism coinvariant module M, analogous property of the injective case hold.

Theorem 5.11. *Let M be χ-automorphism coinvariant. If M is directly finite, then X is directly finite.*

Theorem 5.12. *Let M be χ-automrphism coinvariant. If M is directly finite then $End(M)/J(End(M))$ is unit-regular.*

We can find also an equivalent of Theorem 3.20:

Theorem 5.13. *Let M be χ-automorphism coinvariant. Then the following are equivalent:*

(a) The module M is directly finite.

(b) The module M has the internal cancellation property.

(c) The module M has the cancellation property.

(d) The module M has the substitution property.

(e) The module X is directly finite.

(f) The module X has the internal cancellation property.

(g) The module X has the cancellation property.

(h) The module X has the substitution property.

As for the injective case we have a theorem that states that if M is χ-automorphism coinvariant and X is indecomposable, then M is χ-automorphism coinvariant. We can find also an analogue of Theorem 3.12:

Theorem 5.14. *Let M be χ-automorphism coinvariant and indecomposable. Then the following are equivalent:*

(a) The module M is χ-endomorphism coinvariant.

(b) The module X is indecomposable.

As in the injective case we can say something when M is indecomposable χ-automorphism coinvariant but not χ-endomorphism coinvariant.

Theorem 5.15. *Let M be an indecomposable χ-automorphism coinvariant module that is not χ-endomorphism coinvariant.*

Then $End(M)/J(End(M)) \cong \mathbb{F}_2$ and $End(X)/J(End(X))$ has an homomorphic image isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$.

As for the injective case we can show that not every automorphism-coinvariant module is endomorphism-coinvariant.

Example 5.3.1. Let R and M be respectively the ring and the module given in example 4.1.1. It can be proved that since R is a finite dimensional algebra over $\mathbb{F}_2$, the functors:

$$\mathrm{Hom}_{\mathbb{F}_2}(-, \mathbb{F}_2) : \text{Mod-}R \to R\text{-Mod}$$

and

$$\mathrm{Hom}_{\mathbb{F}_2}(-, \mathbb{F}_2) : R\text{-Mod} \to \text{Mod-}R$$

estabilish a contravariant equivalence between the full subcategories of left and right finitely generated modules over R.

One can show that $\mathrm{Hom}_{\mathbb{F}_2}(E(M), \mathbb{F}_2)$ is the projective cover of $\mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$, and this implies that $\mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$ is an automorphism-coinvariant left R-module. Since $M \cong \mathrm{Hom}_{\mathbb{F}_2}(\mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2), \mathbb{F}_2)$ and

$$E(M) \cong \mathrm{Hom}_{\mathbb{F}_2}(\mathrm{Hom}_{\mathbb{F}_2}(E(M), \mathbb{F}_2), \mathbb{F}_2)$$

we get that $\mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$ is not endomorphism-coinvariant.

5.4 Dual automorphism-invariant modules

In this section, we introduce dual automorphism-invariant modules.

Dual automorphism-invariant modules, under certain conditions, will permit us to generalize Theorem 4.1. In particular, they represent a bridge between pseudo-projective and automorphism-coinvariant modules (we will be basically looking for properties that in the pseudo-injective case are given for free).

We say that a right R-module M is *dual automorphism-invariant* if whenever K_1 and K_2 are small submodules of M, then any epimorphism $\nu : M/K_1 \to M/K_2$ with small kernel lifts to an endomorphism ϕ of M. This means that if we call π_1 and π_2 the natural projections of M over M/K_1 and M/K_2 respectively, then $\nu \circ \pi_1 = \pi_2 \circ \phi$.

We will show in a bit that the endomorphism ϕ of the definiton can be thought as an automorphism (this gives a justification of the name). Before proving the announced result we need a Lemma:

Lemma 5.16. *Let M be a dual automorphism-invariant module and suppose that $\nu : M \to M$ is an epimorphism with small kernel. Then ν is an automorphism.*

Proof. Let $K = Ker(\nu)$. Then $\bar{\nu} : M/K \to M$ (the induced morphism) is an isomorphism, so we can consider its inverse $\bar{\nu}^{-1}$. Since M is a dual automorphism-invariant module, we have that ν lifts to an endomorphism ϕ of M and $\phi(M) + K = M$, but since K is small in M, we get that $M = \phi(M)$. Calling p_k the natural projection of M onto M/K, we have that

$$Ker(\phi) \subseteq Ker(p_K \circ \phi) = \ker(\nu^{-1}) = 0,$$

so ϕ is an isomorphism. Since $x = \bar{v} \circ \bar{v}^{-1}(x) = v \circ \phi(x) = x$ for any $x \in M$ and ϕ is an isomorphism, we deduce that v is an isomorphism. □

Theorem 5.17. *A right R-module M is dual automorphism-invariant if and only if every small epimorphism $\nu : M/K_1 \to M/K_2$, with K_1 and K_2 small submodules of M, lifts to an automorphism of M.*

Proof. One implication is clear. Let M be a dual automorphism-invariant right R-module, let K_1 and K_2 be small submodules of M, and let

$$\nu : M/K_1 \to M/K_2$$

be a small epimorphism with kernel L/K_1 for some submodule L of M. Since K is small we have that L is small, in fact: suppose that $L + M' = M$ for some submodule M' of M. This means that $L/K_1 + (M' + K_1)/K_1 = M$, hence $M' + K_1 = M$, therefore $M' = M$. If we call π_1 the projection of M onto M/K_1 and we set $\lambda = \nu \circ \pi_1$, we get that $Ker(\lambda) = L$, therefore λ is an epimorphism with small kernel. Moreover, since M is dual automorphism-invariant, we obtain that λ lifts to a endomorphism ϕ of M such that $\phi(M) + K_2 = M$. Since K_2 is small we immediatley get that ϕ is surjective, and by Lemma 5.16 we can conclude.

□

A module with no non-zero small submodule is automatically dual automorphism-invariant. Hence, since the Jacobson radical of a module M is the sum of the superfluous right submodules of M, we get that a semiprimitive module is dual automorphism-invariant.

We know that for a commutative ring R, it is equivalent to be Von Neumann regular or such that all the simple right R-modules are injective. In the non-commutative case this equivalence does not always work. This justify the following definition: we say that a ring R is a right V-ring if every simple right R-module over R is injective.

Theorem 5.18. *For a ring R the following are equivalent:*

(a) The ring R is a right V-ring.

(b) Every right R-module M has Jacobson radical equal to zero.

Proof. We just prove that (a)$\Rightarrow$ (b).

We have just to prove that every cyclic submodule xR of M is not contained in the Jacobson radical. Since xR is a finitely generated module, it has for sure a maximal submodule, so in particular there exists an epimorphism $g : xR \to S$, for some simple right R-module S. Since S is injective we can extend g to an epimorphism $h : M \to S$, therefore $Ker(h)$ is a maximal submodule of M not containing x and we can conclude. □

Since every semiprimitive module is dual automorphism-invariant, from Theorem 5.18, we get the following theorem:

Theorem 5.19. *Let R be a rigt V-ring. Then any right R-module is dual automorphism-invariant.*

It is now natural to ask wheter the converse of Theorem 5.19 holds. Before proving that the answer is affirmative, we need a Lemma:

Lemma 5.20. *Let M_1 and M_2 be right R-modules. If $M_1 \oplus M_2$ is dual automorphism-invariant, then any homomorphism $f : M_1 \to M_2/K_2$, with K_2 small in M_2 and $Ker(f)$ small in M_1, lifts to an homomorphism*

$$g : M_1 \to M_2.$$

Proof. Let $\sigma : M \to M/K_2$ be the epimorphism defined by

$$\sigma(m_1 + m_2) = m_1 + (f(m_1) + m_2 + K_2).$$

Since K_2 is small in M_2, and $M_2 \subseteq M$, K_2 is small in M. As M is dual automorphism-invarinat, σ lifts to an isomorphism ν of M. If we consider an element $m_1 \in M_1$, $\nu(m_1) = u_1 + u_2$ for some elements $u_1 \in M_1$ and $u_2 \in M_2$, then $u_1 + u_2 + K_2 = m_1 + (f(m_1) + K_2)$, therefore $f(m_1) = u_2 + K_2$. Let π_2 be the natural projection of M onto M_2. Then $g = \pi_2 \circ \nu_{|M_1} : M_1 \to M_2$ lifts f. □

Theorem 5.21. *A ring R is a right V-ring if and only if every finitely generated right R-module is dual automorphism-invariant.*

Proof. One implication is obvious. Suppose that every finitely generated right R-module is dual-automorphism-invariant. We want to act by contradddiction, so suppose there exists a right simple module S such that S is not injective. This implies that $E(S) \neq S$, hence there exists an $x \in E(S) \setminus S$. Since S is essential in $E(S)$, $xR \cap S \neq 0$ and so S must be contained in xR. As S is essential in $E(S)$, S is essential in xR and this implies that S is small in xR (and so in R). In fact, if there exists a submodule N of

xR such that $N + S = xR$ and $N \neq xR$, we get that $N \cap S = 0$, that is absurd since S is essential in xR. Consider now two submodules N_1 and N_2 of xR such that $N_1 \cap N_2 = 0$. As S is essential in xR and is simple, this means that $N_1 = 0$ or $N_2 = 0$. Therefore, xR is uniform. Let $A = r(x)$. Since $xR \cong R/A$ there exists a right ideal B of R such that $S \cong B/A$ and $A \subset B \subset R$. As $(R/A)/(B/A) \cong R/B$ we can consider the identity homomorphism $1_{R/B} : R/B \to R/B \cong (R/A)/(B/A)$, with $Ker(1_{R/B}) = 0$ small in R/B and B/A small in R/A. So in particular, if we consider the finitely generated module $M = R/B \oplus R/A$, we obtain that M is dual automorphism-invariant by hypotesis, and $1_{R/B}$ satisfies the hypotesis of Lemma 5.20. In particular, this implies that $1_{R/B}$ can be lifted to a morphism $\nu : R/B \to R/A$. Call π_B the projection of R/A on R/B. Since $\pi_B \circ \nu$ is the identity of R/B we obtain that $Im(\nu)$ is a direct summand of xR, which is not possible since R/A is uniform. Therefore, we can conclude that R is a V-ring. □

We would like to extend some results obtained for the injective case to the projective one, but the problem is that not for every ring R, every right R-module has a projective cover. So it is natural to define perfect rings.

We say that a ring R is *right perfect* if every right R-module M has a projective cover.

We want now to define an analogue of the concept of pseudo-injective modules in the projective case. It is then natural to give the following definition: we say that a module M is *pseudo-projective* if every epimorphism $\phi : M \to M/N$, with N a submodule of M, can be lifted to an endomorphism of M. More formally, if p is the projection of M on M/N, then there exists an endomorphism f of M such that $p \circ f = \phi$.

We will see that a pseudo-projective with a projective cover is automorphism-coinvariant. In particular, if R is a right perfect ring, every pseudo-projective is automorphism-coinvariant.

We have now to introduce a Lemma that will be useful in the future.

Lemma 5.22. *Let A and B be right R modules, and let C be a small submodule of A. Furthermore, let $f : A \to B$, $g : A \to B$ be two R-module morphisms such that $g(C) = 0$. Call π the projection of B over $B/f(C)$ and consider $f' = \pi \circ f$, $g' = \pi \circ g$. If $f' = g'$, then $f = g$.*

Proof. By hypotesis, for every $a \in A$, $f(a) + f(C) = g(a) + f(C)$. Hence, $(f - g)(A) \subseteq (f - g)(C)$, therefore $A = C + Ker(f - g)$. Since C si small in A we get that $A = Ker(f - g)$, so $f = g$. □

Theorem 5.23. *A pseudo-projective module is dual autmorphism-invariant.*

Proof. It is straightforward. □

Our next goal is to prove that every pseudo-projective module that admits a projective cover is automorphism-coinvariant. Before doing that we need to prove the following:

Theorem 5.24. *Let P a projective module and let K be a small submodule of P such that $M = P/K$ is dual automorphism-invariant. Then $\sigma(K) = K$ for every automorphism of P, therefore every automorphism of P induces an automorphism of M. This implies in particular that every dual automorphism-invariant module with a projective cover is automorphism-coinvariant.*

Proof. Let $\sigma : P \to P$ be an automorphism and suppose that $\sigma(K) \nsubseteq K$. The induced map $\bar{\sigma} : P/K \to P/(K + \sigma(K))$ has small kernel,

$$Ker(\bar{\sigma}) = (\sigma^{-1}(K) + K)/K.$$

Since M is dual automorphism-invariant, $\bar{\sigma}$ lifts to an automorphism ν of M, hence ν lifts to an automorphism λ of P (λ is surjective since K is small, and a small epimorphism between two projective modules is an isomorphism). Let $\bar{\lambda}$ be the composition of λ with the projection π of P onto P/K and let $\mu : P \to P/K$ be the composition of σ with π. Call π' the natural projection of P/K onto $P/(K+\sigma(K))$. Then $\pi' \circ \bar{\lambda} = \pi' \circ \mu$, therefore, by Lemma 5.22, $\bar{\lambda} = \mu$. But $Ker(\mu) = \sigma^{-1}(K)$ and $Ker(\bar{\lambda}) = K$, we ge that $\sigma^{-1}(K) = K$, hence $\sigma(K) = K$. □

Now we are ready to prove the result we announced at the beginning of the section:

Theorem 5.25. *If P is a projective module and K is superfluous in P, then P/K is dual automorphism-invariant if and only if for every automorphism σ of P we have that $\sigma(K) = K$. In particular, a module M with a projective cover is automorphism-coinvariant if and only if it is dual automorphism-invariant.*

Proof. From Theorem 5.24, it follows that we have just to prove one implication. Suppose that M is automorphism-coinvariant, so $\sigma(K) = K$ for every automorphism of P. Let $\bar{L_1} = L_1/K$ and let $\bar{L_2} = L_2/K$ be two small submodules of M and $\sigma : M/\bar{L_1} \to M/\bar{L_2}$ be a small epimorphism. Then $Ker(\sigma) = \bar{L}/\bar{L_1}$ where $\bar{L} = L/L_1$, with L a submodule of Pcontaing K. Then L is small in M, so it is small in P. Now σ induces an epimorphism $\sigma' : P/L_1 \to P/L_2$ such that for any $x \in P$, $\sigma'(x + L_1) = y + L_2$ if and only if $\sigma(\bar{x} + \bar{L_1}) = \bar{y} + \bar{L_2}$. Now $Ker(\sigma') = L/L_1$ is small in P/L_1 and σ' is an epimorphism. Hence it lifts to an epimorphism ν of P. Then $Ker(\nu) \subseteq L$ and therefore $Ker(\nu)$ is small in P. Then ν is an automorphism of P (Theorem 5.16), so $\nu(K) = K$, hence $\bar{\nu}$ induces an automorphism of M and $\bar{\nu}$ lifts to σ.

□

From Theorem 5.23 and Theorem 5.25, we get immediately the following:

Theorem 5.26. *Let M be a pseudo-projective right R-module and suppose that M admits a pojective cover. Then M is automorphism-coinvariant.*

Corollary 5.27. *Let M be a quasi-projective right R-module that admits a projective cover. Then M is automorphism-coinvariant.*

5.5 Right perfect rings

In this section, we generalize the results given in Section 4.2. The problem is that there exist rings R such that not every right R-module has a projective cover. The natural solution is to consider right perfect rings.

We say that a class χ of right R-modules is a *covering class* if every right R-module admits a χ-cover.

If R is right perfect then the class χ of projective modules is a covering class.

When R is a right perfect ring, we can find some results that play the role of the ones introduced for commutative noetherian rings in the injective case.

Theorem 5.28. *Let R be a right perfect ring and let M be a right R- module. Then a module M is automorphism-coinvariant if and only if it is pseudo-projective.*

Theorem 5.29. *Let R be a right perfect ring, let M be an automorphism-coinvariant module with projective cover $\pi : P \to M$ and assume that M is not endomorphism-coinvariant. Assume furthermore that M is indecomposable and has finite Goldie dimension. Then the following hold:*

(a) The ring $End(M)/J(End(M))$ is isomorphic to $\mathbb{F}_2$.

(b) There exists an $n \geq 2$ and indecomposable modules P_i, with $i = 1, \ldots, n$, two by two non-isomorphic such that $P = \oplus_{i=1}^{n} P_i$ and

$$End(P_i)/J(End(P_i)) \cong \mathbb{F}_2$$

for every $i = 1, \ldots, n$.

In particular, one can obtain an analogue of Corollary 4.18.

Corollary 5.30. *Any automorphism-coinvariant module over a commutative local perfect ring is quasi-projective.*

It is natural then to suppose that the analogy with commutative noetherian rings can be extended furthermore. We can give in fact an analogue of Theorem 4.21:

Theorem 5.31. *Let R be a commutative perfect ring. Then any automorphism-coinvariant module over R is quasi-projective.*

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